Some Useful Numbers, Equations and Identities

\[ \Delta x \cdot \Delta p_x \geq \frac{\hbar}{2} ; \quad \Delta E \cdot \Delta t \geq \frac{\hbar}{2} \]

TDSE: \[ -\frac{\hbar^2}{2m} \frac{\partial^2 \psi(x, t)}{\partial x^2} + U(x) \psi(x, t) = i\hbar \frac{\partial \psi(x, t)}{\partial t} \]

TISE: \[ -\frac{\hbar^2}{2m} \frac{\partial^2 \psi(x)}{\partial x^2} + U(x) \psi(x) = E \psi(x) \]

Operator: \[ \hat{p} = \frac{\hbar}{i} \frac{\partial}{\partial x} , \quad [\hat{p}^2] = -\hbar^2 \frac{\partial^2}{\partial x^2} , \quad [\hat{E}] = i\hbar \frac{\partial}{\partial t} \]

Uncertainty in observable \( Q \): \[ \Delta Q = \sqrt{\langle Q^2 \rangle - \langle Q \rangle^2} \]

Expectation Value \( \langle Q \rangle \): \[ \int \psi^*(x) \langle Q \rangle \psi(x) \, dx \]

\[ \int_0^{+\infty} e^{-ax^2} \, dx = \frac{1}{2} \sqrt{\frac{\pi}{a}} , \quad a > 0 \]

\[ \int_0^{+\infty} x^2 e^{-ax^2} \, dx = \frac{1}{4a} \sqrt{\frac{\pi}{a}} , \quad a > 0 \]

\[ \int_0^{+\infty} x^{2n} e^{-ax^2} \, dx = \frac{1.3...)(2n-1)}{2^{n+1}} \sqrt{\frac{\pi}{a^{2n+1}}} , \quad a > 0 \]

\[ \int_0^{+\infty} x^{2n+1} e^{-ax^2} \, dx = \frac{n!}{2a^{n+1}} , \quad a > 0 \]

For a Quantum oscillator under \( U(x) = \frac{1}{2} m \omega^2 x^2 \) the ground state wavefunction is: \( \psi_0(x) = \left( \frac{m \omega}{\pi \hbar} \right)^{1/4} e^{-m \omega x^2 / 2\hbar} \) and energy is \( E = \frac{1}{2} \hbar \omega \)

Please consult the proctor if you don’t understand any part of the questions
Problem 1: Déjà Vu all over again!  [12 pts]
(a) Calculate $<x>, <x^2>$ and $\Delta x$ for a quantum oscillator of mass $m$ and angular frequency $\omega$ in its ground state. (b) What is the average potential energy for this system. Calculate (c) the average momentum $<p>$, (d) the average kinetic energy $<KE>$, (e) $<p^2>$ and (f) the product $\Delta x.\Delta p$.

Problem 2: A Step in the Right Direction [8 pts]
A particle is incident on a potential step characterized by
$$U(x) = \begin{cases} 0 & \text{for } x < 0 \\ U & \text{for } x > 0 \end{cases}$$
with a kinetic energy $E < U$. It is described by the wavefunction
$$\psi(x) = \frac{1}{2} \left\{ (1+i)e^{ikx} + (1-i)e^{-ikx} \right\} \text{ for } x \leq 0$$
$$\psi(x) = e^{-kx} \text{ for } x \geq 0$$
(a) verify by direct calculation that the reflection coefficient $R$ is unity (b) How must $k$ be related to $E$ in order for $\psi(x)$ to solve Schrodinger equation in the region to the left of the step ($x \leq 0$) and to the right of the step. Calculate the relation between energy $E$ and the potential $U$. 