1. Problem 1

a) $\langle x \rangle = 0$ by symmetry. You could also do the integral $\int_{-\infty}^{\infty} xe^{-ax^2}$, but x is odd and e^{-ax^2} is even, so when you integrate over a symmetric interval, you get zero.

$$\begin{aligned}
\langle x^2 \rangle &= \int_{-\infty}^{\infty} \sqrt{\frac{a}{\pi}} e^{-ax^2} x^2 dx \\
\text{Where } a &= \frac{m\omega}{\hbar} \\
\langle x^2 \rangle &= \sqrt{\frac{a}{\pi}} \int_{-\infty}^{\infty} \frac{-\partial}{\partial a} e^{-ax^2} dx \\
&= \sqrt{\frac{a}{\pi}} \frac{-\partial}{\partial a} \int_{-\infty}^{\infty} e^{-ax^2} dx \\
&= \sqrt{\frac{a}{\pi}} \frac{-\partial}{\partial a} \sqrt{\frac{\pi}{a}} \\
&= \frac{1}{2a} = \frac{\hbar}{2m\omega}
\end{aligned} \tag{1.1}$$

$$\begin{aligned}
\Delta x &= \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \\
&= \sqrt{\frac{\hbar}{2m\omega}}
\end{aligned} \tag{1.2}$$

b) $PE = \frac{1}{2} m\omega^2 x^2$, so:

$$\begin{aligned}
\langle PE \rangle &= \langle \tfrac{1}{2} m\omega^2 x^2 \rangle \\
&= \tfrac{1}{2} m\omega^2 \langle x^2 \rangle \\
&= \frac{1}{4} \hbar\omega
\end{aligned} \tag{1.3}$$

c) $\langle p \rangle = 0$ by symmetry. You could also do the integral, but it would vanish for the same reason the integral in part a) did.

d) Since $E = KE + PE = \frac{1}{2}\hbar\omega$ for the ground state, $\frac{1}{2}\hbar\omega = \langle KE \rangle + \langle PE \rangle = \langle KE \rangle + \frac{1}{4}\hbar\omega$. So we have that $\langle KE \rangle = \frac{1}{4}\hbar\omega$

e) Notice $KE = \frac{p^2}{2m}$, therefore $\langle KE \rangle = \frac{1}{4}\hbar\omega = \frac{\langle p^2 \rangle}{2m}$. This gives $\langle p^2 \rangle = \frac{1}{2} m\omega\hbar$. You could also do the integral $\int_{-\infty}^{\infty} (-\hbar^2 \frac{\partial^2}{\partial x^2}) e^{-ax^2}$, but you will get the same answer and it will take a lot longer.

f)

$$
\begin{aligned}
\Delta p &= \sqrt{\langle p^2 \rangle - \langle p \rangle^2} \\
&= \sqrt{\frac{m\hbar\omega}{2}} \\
\Delta p \Delta x &= \sqrt{\frac{m\hbar\omega}{2}}\sqrt{\frac{\hbar}{2m\omega}} \\
&= \frac{\hbar}{2}
\end{aligned} \tag{1.4}
$$

2. Problem 2

a)

$$
\begin{aligned}
R &= \frac{|\frac{1}{2}(1-i)|^2}{|\frac{1}{2}(1+i)|^2} \\
&= \frac{(1-i)(1+i)}{(1+i)(1-i)} \\
&= 1
\end{aligned} \tag{2.1}
$$

b) In the region where $x < 0$, we can plug into the Schrodinger equation and we get: $E\psi = \frac{\hbar^2 k^2}{2m}\psi$. So we find that $k^2 = \frac{2mE}{\hbar^2}$. E will be the same in the region where $x > 0$, so if the k is the same in both regions, the same expression for k^2 must hold when $x > 0$.

We can also plug the $x > 0$ part into the Schodinger Equation, where we will find that $k^2 = \frac{2m(U-E)}{\hbar^2}$. Comparing this with the formula for k^2 above, we find that $E = \frac{1}{2}U$.