4E Week 7 HW

7: 11, 15, 17, 18, 19, 21, 22, 23, 25, 26, 28, 66, 68

7.11 a) $\omega = 2\pi \cdot \frac{735}{60} \frac{\text{rev}}{\text{s}}$

$$L = I\omega = \boxed{7.7 \times 10^{-4} \frac{\text{kg}\cdot\text{m}^2}{\text{s}^2}}$$

b) $L^2 = \hbar^2 \ell(\ell+1) \Rightarrow \ell(\ell+1) = \frac{L^2}{\hbar^2}$

So $\ell^2 + \ell - \frac{L^2}{\hbar^2} = 0$

$$\Rightarrow \ell = \frac{-1 + \sqrt{1 + 4L^2/\hbar^2}}{2} \approx \frac{L}{\hbar}$$

So $\boxed{\ell \approx 7.3 \times 10^{30}}$

7.15) $\vec{L} = \vec{r} \times \vec{p} \Rightarrow \frac{\partial \vec{L}}{\partial t} = \frac{\partial \vec{r}}{\partial t} \times \vec{p} + \vec{r} \times \frac{\partial \vec{p}}{\partial t}$.

Now, $\frac{\partial \vec{r}}{\partial t} \times \vec{p} = 0$ b/c they're parallel.

And $\frac{d\vec{p}}{dt} = \vec{F} = -\vec{\nabla} V(r) = -\hat{r} \frac{\partial V}{\partial r}$, so this is parallel to $\vec{r}$.

$\therefore \vec{r} \times \frac{d\vec{p}}{dt} = 0 \Rightarrow \boxed{\frac{\partial \vec{L}}{\partial t} = 0}$.

17] Gf a] $\boxed{n=6,\ \ell=3}$

b] $E_6 = \frac{-13.6\,eV}{6^2} = \boxed{-0.38\,eV}$

c] $L = \hbar\sqrt{\ell(\ell+1)} = \boxed{2\sqrt{3}\,\hbar}$

d] $\boxed{L_z = -3\hbar, -2\hbar, -\hbar, 0, \hbar, 2\hbar, 3\hbar}$

18] $R_{30}(r) = \frac{2}{3\sqrt{3a_0^3}}\left(1 - \frac{2r}{3a_0} + \frac{2r^2}{27a_0^2}\right)e^{-r/3a_0}$

So $R_{30}(r) = 0$ when (defining $u \equiv r/a_0$)

$1 - \frac{2}{3}u + \frac{2}{27}u^2 = 0 \Rightarrow 27 - 18u + 2u^2 = 0$

i.e. at $u = 1.9, 7.1$

$\Rightarrow \boxed{r = 1.9a_0,\ 7.1a_0}$

19] $n=1,\ \ell=m=0.\ \therefore \psi_{100} = R_{10}(r)Y_{00}(\theta,\phi)$

$= \frac{2}{a_0^{3/2}}e^{-r/a_0} \cdot \frac{1}{2\pi^{1/2}}$

$= \frac{1}{\pi^{1/2}a_0^{3/2}}e^{-r/a_0}$

So $\boxed{\psi(a_0) = \frac{1}{\pi^{1/2} a_0^{3/2} e}}$

$$\boxed{\psi^2(a_0) = \frac{1}{\pi a_0^3 e^2}}$$

$$P(r) = |rR(r)|^2_{4\pi} = \frac{4r^2}{a_0^3} e^{-2r/a_0} \cdot 4\pi$$

So $\boxed{P(a_0) = \frac{4^2\pi}{a_0} e^{-2}}$

21] $P(r) = Cr^2 e^{-2Zr/a_0}$

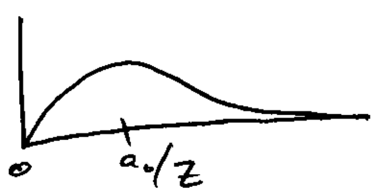
$$\frac{\partial P}{\partial r} = C\left[2r e^{-2Zr/a_0} - r^2 \cdot \frac{2Z}{a_0} e^{-2Zr/a_0}\right] = 0$$

$$\Rightarrow \quad 2r = \frac{2r^2 Z}{a_0} \Rightarrow r = (Z/a_0)^{-1} = \frac{a_0}{Z}$$

or

$$r = 0$$

$\boxed{r = a_0/Z}$ is clearly the max:



22] $\psi_{210}(r,\theta,\phi) = C_{210}\dfrac{Zr}{a_0}e^{-Zr/a_0 \cdot 2}\cos\theta$

Normalize: $\displaystyle\int_0^\infty dr\int_0^\pi d\theta\int_0^{2\pi} d\phi\; r^2\sin\theta\,|\psi_{210}|^2 = 1$

We can break this up into 3 integrals:

ϕ: $\displaystyle\int_0^{2\pi} d\phi = 2\pi$. ✓

θ: $\displaystyle\int_0^\pi d\theta\,\sin\theta\cos^2\theta =$ $\quad u=\cos\theta$, $du = -\sin\theta\, d\theta$

$$= \int_{-1}^{1} du\cdot u^2 = \boxed{\frac{2}{3}}$$

r: $\displaystyle\int_0^\infty dr\cdot r^2\cdot r^2 e^{-Zr/a_0} = \left(\frac{a_0}{Z}\right)^5\int_0^\infty du\, u^4 e^{-u} = \overset{24}{4!}\left(\frac{a_0}{Z}\right)^5$

So: $C_{210}^2\cdot\left(\dfrac{Z}{a_0}\right)^2\cdot 2\pi\cdot\dfrac{2}{3}\cdot 4!\left(\dfrac{a_0}{Z}\right)^5 = 1$

$$C_{210}^2 = \left(\frac{Z}{a_0}\right)^3\cdot\frac{3}{96\pi}$$

$$\Rightarrow \boxed{C_{210} = \left(\frac{Z}{a_0}\right)^{3/2}\left(\frac{1}{32\pi}\right)^{1/2}}$$

23] $\psi_{200} = R_{20} Y_{00}$

$$= \frac{1}{\sqrt{2a_0^3}}\left(1 - \frac{r}{2a_0}\right)e^{-r/2a_0} \cdot \frac{1}{\sqrt{4\pi}}.$$

$$|\psi|^2 = \frac{1}{8\pi a_0^3}\left(1 - \frac{r}{2a_0}\right)^2 e^{-r/a_0}$$

Now, to find it in $\Delta r = 0.02a_0$, just use

$$\text{Prob} = \int dr\, r^2 |\psi|^2 \cdot 4\pi = 4\pi r^2 |\psi|^2 \Delta r \Big|_{r=a_0,\, \Delta r = 0.02a_0}$$

$$= \frac{1}{2a_0^3} \cdot a_0^2 \cdot \frac{1}{4} e^{-1} (0.02a_0)$$

$$= \frac{0.02}{8} e^{-1} = \boxed{9.1 \times 10^{-4}}$$

b] Same as above, but plug in $r = 2a_0$:

$$\frac{1}{2a_0^3}(2a_0)^2 \cdot 0 \cdot e^{-2} \Delta r = \boxed{0}$$

25] $\psi_{200} = R_{20} Y_{00}$

$$= \frac{1}{\sqrt{8\pi a_0^3}} \left(1 - \frac{r}{2a_0}\right) e^{-r/2a_0}.$$

So $\psi_{200}(a_0) = \boxed{\frac{1}{2\sqrt{8\pi a_0^3}} e^{-1/2}}$

b] $\psi_{200}^2(a_0) =$ This squared

c] $P(r) = 4\pi r^2 |R(r)|^2$

$$= 4\pi r^2 \frac{1}{2a_0^3} \left(1 - \frac{r}{2a_0}\right)^2 e^{-r/a_0}$$

$$P(a_0) = \frac{4\pi \cdot a_0^2}{2a_0^3} \cdot \frac{1}{4} e^{-1} = \boxed{\frac{\pi}{2a_0 e}}$$

26] $r^2 |R(r)|^2 = \left(\frac{r}{a_0}\right)^4 e^{-r/a_0} \sim P(r)$

$$\frac{d}{dr}\left(r^2 |R|^2\right) = \frac{4r^3}{a_0^4} e^{-r/a_0} + \left(\frac{r}{a_0}\right)^4 \left(-\frac{1}{a_0}\right) e^{-r/a_0} = 0$$

So $4r^3 - r^4/a_0 = 0$

$$\Rightarrow r^4 = 4a_0 r^3 \Rightarrow \boxed{r = 4a_0}$$

7-28 | Just Plug it in!

7-66 | $|\vec{L}| = \hbar\sqrt{\ell(\ell+1)}$, $L_z = m_\ell \hbar$.

The minimum angle happens when m_ℓ is biggest, $m_\ell = \ell$.

This angle is 

$\Rightarrow \cos\theta = \dfrac{|L_z|}{|L|}$

So $\boxed{\cos\theta = \dfrac{\ell\hbar}{\hbar\sqrt{\ell(\ell+1)}} = \sqrt{\dfrac{\ell}{\ell+1}}}$.

For large ℓ, $\dfrac{\ell}{\ell+1} \simeq 1$, so $\theta \simeq 0$. Expand:

$\cos\theta \simeq 1 - \dfrac{\theta^2}{2}$, $\left(\dfrac{\ell}{\ell+1}\right)^{1/2} = \left(\dfrac{1}{1+1/\ell}\right)^{1/2} \simeq 1 - \dfrac{1}{2\ell}$

$\therefore \dfrac{\theta^2}{2} \simeq \dfrac{1}{2\ell} \Rightarrow \boxed{\theta \simeq \dfrac{1}{\ell^{1/2}}}$

68] $\psi_{100} = \frac{2}{a_0^{3/2}} e^{-r/a_0} \cdot \frac{1}{\sqrt{4\pi}}$

So $\int |\psi_{100}|^2 r^2 \sin\theta = \int_0^{R_0} r^2 \cdot \frac{4}{a_0^3}\, dr$, using $e^{-r/a_0} \simeq 1$.

$$\Rightarrow \quad = \frac{4}{3}\frac{R_0^3}{a_0^3} = \frac{4}{3}\left(10^{-5.2}\right)^3 \approx \boxed{10^{-14}}.$$