1. 6-32

Since V(x) is symmetric, we know that $\langle x \rangle$ should be zero, but we can check this. Since $\psi_0 = A_0 e^{\frac{-m\omega x^2}{2\hbar}}$ we have:

$$\begin{aligned} \langle x \rangle &= \int_{-\infty}^{\infty} A_0^2 x e^{\frac{-m\omega x^2}{\hbar}} \\ &= \int A_0^2 x e^{-\alpha x^2} \end{aligned} \tag{1.1}$$

Since x is and odd function and $e^{-\alpha x^2}$ is an even function, when we integrate over a symmetric interval we get zero. So as expected, $\langle x \rangle = 0$. Here $\alpha = m\omega^2/\hbar$. Notice $A_0 = (\frac{\alpha}{\pi})^{1/4}$. For $\langle x^2 \rangle$:

$$\begin{aligned} \langle x^2 \rangle = \int A_0^2 x^2 e^{-\alpha x^2} &= \int \sqrt{\frac{\alpha}{\pi}}(-\frac{\partial}{\partial \alpha})e^{-\alpha x^2} \\ &= -\sqrt{\frac{\alpha}{\pi}}\frac{\partial}{\partial \alpha}(\int e^{-\alpha x^2}) \\ &= -\sqrt{\frac{\alpha}{\pi}}\frac{\partial}{\partial \alpha}(\sqrt{\frac{\pi}{\alpha}}) \\ &= \tfrac{1}{2}\sqrt{\alpha}\alpha^{-3/2} \\ &= \frac{1}{2\alpha} = \tfrac{1}{2}\frac{\hbar}{m\omega} \end{aligned} \tag{1.2}$$

Here I used a trick which is sometimes convenient, but you can look this integral up if you need to.

2. 6-33

For the harmonic oscillator we know that $E = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 = (n + \frac{1}{2})\hbar\omega$. For the ground state $n = 0$. Solving for x^2 we have:

$$\langle x^2 \rangle = \frac{2}{m\omega^2}(\tfrac{1}{2}\hbar\omega - \frac{p^2}{2m})$$

Comparing to 6-32

$$\begin{aligned} \frac{2}{m\omega^2}(\tfrac{1}{2}\hbar\omega - \frac{p^2}{2m}) &= \tfrac{1}{2}\frac{\hbar}{m\omega} \\ \langle 1 - p^2/m\hbar\omega \rangle &= \tfrac{1}{2} \\ \langle p^2 \rangle & \\ &= \tfrac{1}{2}m\hbar\omega \end{aligned} \tag{2.1}$$

Phys 4E Week 5 HW

Ch 6: 28, 29, 30, 31, 36, 37, 41, 42, 56, 58, 61, 62

For 28-31 See last week's solutions.

36 a) $\psi_1(x) = C_1 x e^{-\frac{m\omega x^2}{2\hbar}}$

So $\int_{-\infty}^{\infty} C_1^2 x^2 e^{-m\omega x^2/\hbar} dx = 1$. But $\int_{-\infty}^{\infty} x^2 e^{-ax^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{a^3}}$

So $\int_{-\infty}^{\infty} x^2 e^{-m\omega x^2/\hbar} dx = \frac{1}{2}\sqrt{\pi}\left(\frac{\hbar}{m\omega}\right)^{3/2} = \left(\frac{\pi\hbar^3}{4m^3\omega^3}\right)^{1/2}$

$\therefore C_1^2 = \left(\frac{4m^3\omega^3}{\pi\hbar^3}\right)^{1/2} \Rightarrow \boxed{C_1 = \left(\frac{4m^3\omega^3}{\pi\hbar^3}\right)^{1/4}}$

b) $\langle x \rangle = \int_{-\infty}^{\infty} C_1^2 x^3 e^{-m\omega x^2/\hbar} dx = 0$ b/c $x^3 e^{-x^2}$ is odd.
$(f(-x) = -f(x))$

c) $\langle x^2 \rangle = \int_{-\infty}^{\infty} C_1^2 x^4 e^{-m\omega x^2/\hbar} dx = C_1^2 \int_{-\infty}^{\infty} x^4 e^{-m\omega x^2/\hbar} dx$

But $\int_{-\infty}^{\infty} x^4 e^{-ax^2} dx = \frac{3}{4}\sqrt{\frac{\pi}{a^5}}$, so $\langle x^2 \rangle = C_1^2 \frac{3}{4}\sqrt{\pi}\left(\frac{\hbar}{m\omega}\right)^{5/2}$

$\therefore \langle x^2 \rangle = 2\left(\frac{m^3\omega^3}{\pi\hbar^3}\right)^{1/2} \cdot \frac{3}{4}\sqrt{\pi}\left(\frac{\hbar^5}{m^5\omega^5}\right)^{1/2} = \boxed{\frac{3}{2}\frac{\hbar}{m\omega}}$

d] $V(x) = \frac{1}{2} m\omega^2 x^2$, so $\langle V(x) \rangle = \frac{1}{2} m\omega^2 \langle x^2 \rangle$

$$= \frac{1}{2} m\omega^2 \cdot \frac{3}{2} \frac{\hbar}{m\omega} = \boxed{\frac{3}{4} \hbar\omega}$$

37] a] $\Delta p = \frac{\hbar}{2\Delta x} = \boxed{\frac{\hbar}{4A}}$

b] $\Rightarrow KE = \frac{p^2}{2m} = \frac{\hbar^2}{32 m A^2}$.

But $E = \frac{1}{2} k A^2 = \frac{1}{2} \hbar\omega \Rightarrow A^2 = \frac{\hbar\omega}{k}$

and since $\omega = \sqrt{\frac{k}{m}}$, $k = m\omega^2 \Rightarrow A^2 = \frac{\hbar\omega}{m\omega^2} = \frac{\hbar}{m\omega}$

$$\Rightarrow KE = \frac{\hbar^2 m\omega}{32 m \hbar} = \frac{1}{32} \hbar\omega.$$

The ground state total energy is $\frac{1}{2} \hbar\omega$, so the H.U.P. estimate is off by a factor of 16.

$\langle KE \rangle = \langle E \rangle - \langle V \rangle$. For the ground state,

$\langle V \rangle = \frac{1}{2} m\omega^2 \langle x^2 \rangle = \frac{1}{2} m\omega^2 \cdot \frac{1}{2} \frac{\hbar}{m\omega} = \frac{1}{4} \hbar\omega$ (similar to Prob. 36).

$\Rightarrow \langle KE \rangle = \frac{1}{2} \hbar\omega - \frac{1}{4} \hbar\omega = \frac{1}{4} \hbar\omega$. So we're off by a factor of 8.

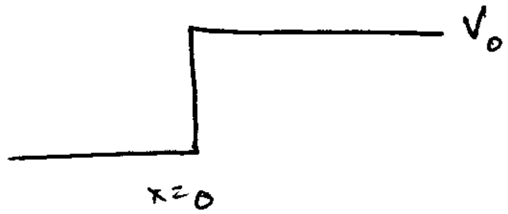
3. 6-40

Ignoring the constants we have:

$$\begin{aligned}\int e^{-(\alpha/2)x^2} x e^{-(\alpha/2)x^2} &= \int x e^{-\alpha x^2} \\ &= 0\end{aligned} \tag{3.1}$$

Again, x is odd and $e^{-\alpha x^2}$ is even, so when we integrate we get 0.

41]



a] Sch. Eqn. is $-\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + V_0\psi = E\psi$ for $x>0$

$$\Rightarrow \frac{\partial^2\psi}{\partial x^2} = -\frac{2m(E-V_0)}{\hbar^2}\psi$$

$$\text{So } k_2^2 = \frac{2m(E-V_0)}{\hbar^2} = \frac{2mV_0}{\hbar^2}$$

$$\text{But } k_1^2 = \frac{2m\cdot 2V_0}{\hbar^2} = \frac{4mV_0}{\hbar^2} \Rightarrow k_2^2 = \frac{k_1^2}{2} \Rightarrow \boxed{k_2 = \frac{k_1}{\sqrt{2}}}$$

b] For $x<0$, $\psi_I = Ae^{ik_1x} + Be^{-ik_1x}$

$x>0$, $\psi_{II} = Ce^{ik_2x}$

Matching: $\psi_I(0) = \psi_{II}(0)$: $A+B=C$

$\psi_I'(0) = \psi_{II}'(0)$: $k_1(A-B) = k_2C$

$$R = \frac{|B|^2}{|A|^2} = \left(\frac{k_1-k_2}{k_1+k_2}\right)^2 = \left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)^2 = \boxed{2.94\%}$$

c] $T = 1-R = \boxed{97.06\%}$

d] $T(1{,}000{,}000) = \boxed{970{,}600}$

Classically, they all would keep going.

42]

a] Now $K_2^2 = \frac{2m(2V_0 + V_0)}{\hbar^2} = \frac{6mV_0}{\hbar^2}$

So $K_2^2 = \frac{3}{2}K_1^2 \Rightarrow \boxed{K_2 = \sqrt{\frac{3}{2}}\,K_1}$

b] $R = \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2 = \left(\frac{1 - \sqrt{3/2}}{1 + \sqrt{3/2}}\right)^2 = \boxed{1.02\%}$

c] $T = 1 - R = \boxed{98.98\%}$

d] $\boxed{989{,}800}$. Classically, they all keep going.

56] a] $T \simeq 16\frac{E}{V_0}\left(1 - \frac{E}{V_0}\right)e^{-2\alpha a}$

$E = 10\,eV$, $V_0 = 25\,eV$

~~$\alpha = [2m(E - V_0)$~~ $\alpha = \left[\frac{2m(V_0 - E)}{\hbar^2}\right]^{1/2}$, $a = 1\,nm$.

So $\alpha a = \left[\frac{2mc^2(V_0 - E)}{(\hbar c)^2}\right]^{1/2} a = \left[2(511\times10^3\,eV)(15\,eV)\right]^{1/2}\frac{a}{197.33\,eV\cdot nm}$

So $\alpha a = (19.8)\dfrac{a}{nm}$.

$$16\left(\frac{E}{V_0}\right)\left(1-\frac{E}{V_0}\right) = 3.84$$

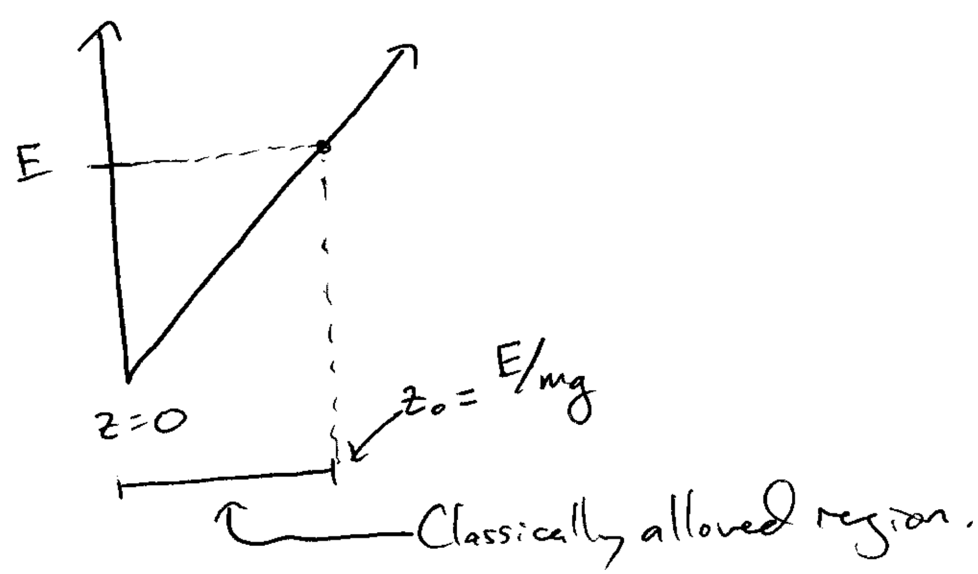
So $T = 3.84\, e^{-39.6a} = \boxed{2.5\times10^{-16}}$

b]

For $a = 0.1\,nm$,

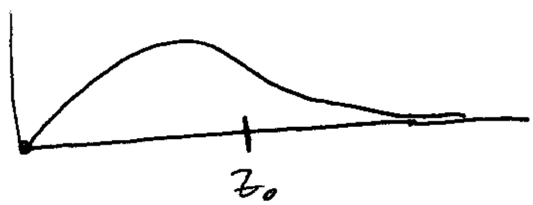
$T = 3.84\, e^{-3.96} = \boxed{7.3\times10^{-2}}$ Much bigger!!

58]

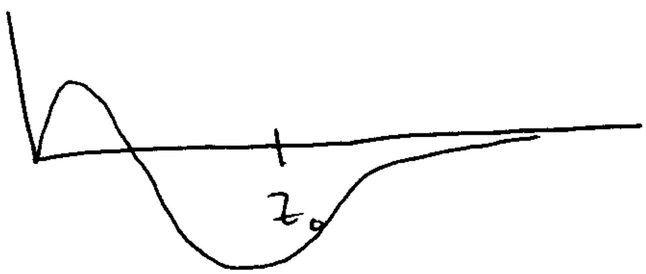


As z approaches the disallowed region, the "wavenumber" will decrease: Treating $k \approx \left(\frac{2m(E-V)}{\hbar^2}\right)$, as V increases, k decreases $\rightarrow$ λ increases:

Ground state: One bump in well, then decay:

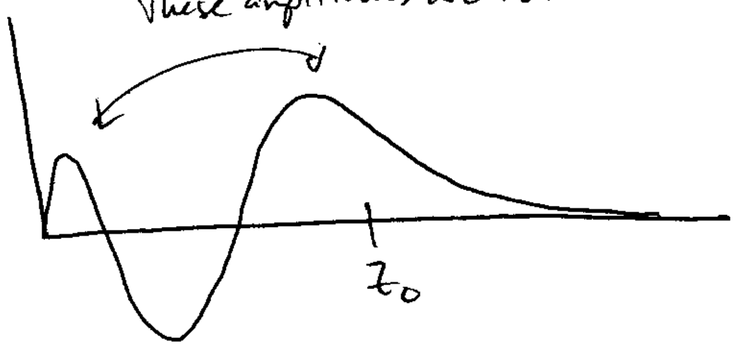


1st excited state: 2 bumps, λ increasing as $z \to z_0$, then decay.



2nd excited state: 3 bumps, then decay:

These amplitudes are not to scale.



61] 6-74 see:

$$\psi_{I} = Ae^{ik_1x} + Be^{-ik_1x}$$

$$\psi_{II} = Ce^{-\alpha x} + De^{\alpha x}$$

$$\psi_{III} = Fe^{ik_1x} + Ge^{-ik_1x}$$

Set $G=0$ for incident from left.

$\therefore$ Matching: $\psi_I(0) = \psi_{II}(0)$: $A + B = C + D$ ①

$\psi_{II}(a) = \psi_{III}(a)$: $Ce^{-\alpha a} + De^{\alpha a} = Fe^{ik_1 a}$ ②

$\psi_I'(0) = \psi_{II}'(0)$: $k_1(A - B) = \alpha(-C + D)$ ③

$\psi_{II}'(a) = \psi_{III}'(a)$: $\alpha(-Ce^{-\alpha a} + De^{+\alpha a}) = +ik_1 Fe^{ik_1 a}$ ④

5 unknowns, 4 eqns. Now solve them with the power of algebra, and it gives the answer (I'm not gonna do it).

$$T = \left[1 + \frac{\sinh^2 \alpha a}{4\frac{E}{V_0}\left(1 - \frac{E}{V_0}\right)}\right]^{-1}$$

b] If $\alpha a >> 1$, $\sinh^2 \alpha a = \left(\frac{e^{+\alpha a} - e^{-\alpha a}}{2}\right)^2 \approx \left(\frac{e^{\alpha a}}{2}\right)^2 = \frac{e^{2\alpha a}}{4}$

$$\text{So } T = \left[1 + \frac{e^{2\alpha a}}{16\frac{E}{V_0}\left(1 - \frac{E}{V_0}\right)}\right]^{-1} = 16\frac{E}{V_0}\left(1 - \frac{E}{V_0}\right)e^{-2\alpha a} \frac{1}{1 + \frac{16\frac{E}{V_0}\left(1-\frac{E}{V_0}\right)}{e^{2\alpha a}}}$$

$$\approx \boxed{16\frac{E}{V_0}\left(1 - \frac{E}{V_0}\right)e^{-2\alpha a}}$$

62] $\psi_{II} = Ce^{-\alpha x}$; $\alpha^2 = 2m(V_0 - E)/\hbar^2$

$\psi_I = Ae^{ikx} + Be^{-ikx}$; $k^2 = 2mE/\hbar^2$

Matching: $A + B = C$, $ik(A-B) = -\alpha C$.

$\Rightarrow A - B = \frac{i\alpha}{k} C$

$\Rightarrow 2A = \left(1 + \frac{i\alpha}{k}\right) C$

$\Rightarrow C = \frac{2A}{\left(1 + \frac{i\alpha}{k}\right)} \Rightarrow |C|^2 = \frac{4|A|^2}{\left(1 + \frac{\alpha^2}{k^2}\right)}$

$= \frac{4|A|^2}{1 + \frac{V_0 - E}{E}}$

$= \frac{4}{1 + \frac{V_0 - E}{E}} \quad (|A|^2 = 1)$.

So $\psi_{II} = 2e^{-\alpha x} \Rightarrow |\psi|^2 = 4e^{-2\alpha x}$.

But $\alpha = \left(\frac{2mc^2 \cdot 20eV}{\hbar^2 c^2}\right)^{1/2} = \left(\frac{2 \cdot 938 MeV \cdot 20eV}{(197.3 MeV \cdot fm)^2}\right)^{1/2} = \frac{9.8 \times 10^{-4}}{fm}$.

Graph: $x = 1\text{ fm}$: $|4|^2 = 3.99$

2 fm 3.984

3 fm 3.976

4 fm 3.968

5 fm 3.960