4E Week 4 HW

Ch. 6: 3, 12, 13, 14, 18, 19, 20, 25, 28, 29, 30, 31

3] We know $E\psi = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + V\psi$, so just solve:

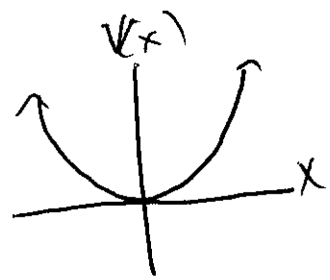
$$\frac{\partial \psi}{\partial x} = \frac{\partial}{\partial x}\left(Ae^{-x^2/2L^2}\right) = Ae^{-x^2/2L^2}\left(\frac{-2x}{2L^2}\right) = -\frac{A}{L^2}xe^{-x^2/2L^2}$$

$$\text{So } \frac{\partial^2 \psi}{\partial x^2} = -\frac{A}{L^2}\left[e^{-x^2/2L^2} - \frac{x^2}{L^2}e^{-x^2/2L^2}\right]$$

$$= -\frac{A}{L^2}\left(1 - \frac{x^2}{L^2}\right)e^{-x^2/2L^2}$$

$$\text{So } +\frac{\hbar^2}{2mL^2} = +\frac{1}{L^2}\left(1 - \frac{x^2}{L^2}\right)\frac{\hbar^2}{2m} + V(x)$$

$$\Rightarrow \boxed{V(x) = \frac{\hbar^2}{2mL^4}x^2}$$



b] This is a harmonic oscillator potential, $V = \frac{1}{2}kx^2$

with $k = \frac{\hbar^2}{mL^4}$

12] $E = \frac{\hbar^2 \pi^2 n^2}{2mL^2} = \frac{p^2}{2m}$.

$p = (10^{-9}\,kg)(10^{-3}\,m/s) = 10^{-12}\,\frac{kg \cdot m}{s}$

So $E = 5 \times 10^{-16}\,J = \frac{\hbar^2 \pi^2 n^2}{2mL^2}$

$\Rightarrow n = \left[\frac{E \cdot 2mL^2}{\hbar^2 \pi^2}\right]^{1/2} = \boxed{3 \times 10^{19}}$ Whoa!

13] a] $\Delta x = 10^{-4} \cdot L = \boxed{10^{-6}\,m}$

$\Delta p = 10^{-4}\,p = \boxed{10^{-16}\,\frac{kg \cdot m}{s}}$

b] $\frac{\Delta x \Delta p}{\hbar} = \boxed{9 \times 10^{11}}$

14] a] Say $\Delta x = \frac{L}{2}$. Then $\Delta p = \frac{\hbar}{2\Delta x} = \frac{\hbar}{L}$.

$E = \frac{p^2}{2m} \simeq \frac{(\Delta p)^2}{2m} = \boxed{\frac{\hbar^2}{2mL^2}}$

b] $E_1 = \boxed{\frac{\hbar^2 \pi^2}{2mL^2}}$ $\Rightarrow$ A factor of π^2 bigger than part a.

18] a] $\psi_5 = \sqrt{\frac{2}{L}} \sin\left(\frac{5\pi x}{L}\right)$.

So prob. of finding it in between $x = 0.2L$ & $x = 0.4L$

is $\displaystyle\int_{0.2L}^{0.4L} |\psi_5|^2 dx = \int_{0.2L}^{0.4L} \left(\frac{2}{L}\right) \sin^2\left(\frac{5\pi x}{L}\right) dx$

$$= \frac{2}{L} \int_{0.2L}^{0.4L} \sin^2\left[\frac{5\pi x}{L}\right] dx$$

use $u = \frac{5\pi x}{L}$, $du = \frac{5\pi}{L} dx$

$$\Rightarrow \frac{2}{L} \int_{\pi}^{2\pi} \frac{L}{5\pi} \sin^2 u \, du$$

$$= \frac{2}{5\pi}\left[\frac{u}{2} - \frac{\sin 2u}{4}\right|_{\pi}^{2\pi} = \frac{2}{5\pi}\frac{\pi}{2} = \boxed{\frac{1}{5}}$$

b] We could do an integral, but it's a small enough interval that we can estimate it by

$$|\psi(\tfrac{L}{2})|^2 \Delta x = |\psi(\tfrac{L}{2})|^2 (0.01L)$$

$$= \frac{2}{\cancel{L}} \sin^2\left(\frac{5\pi \cancel{L}}{2\cancel{L}}\right)(0.01\cancel{L})$$

$$= \boxed{0.02}$$

19] $E_1 = \dfrac{\hbar^2\pi^2}{2mL^2}$

So for electron, $E = \dfrac{(\hbar c)^2\pi^2}{2mc^2L^2} = \dfrac{(197.3\,\text{MeV}\cdot\text{fm})^2\pi^2}{2(.511\,\text{MeV})\,10\text{fm}^2}$

$$= \boxed{3.76\times10^3\ \text{MeV}}$$

proton: $E = \dfrac{(197.3\,\text{MeV}\cdot\text{fm})^2\pi^2}{2(938\,\text{MeV})(10\text{fm})^2} = \boxed{2.05\,\text{MeV}}$

b] $\Delta E = 3E_1$ so: electron: $\boxed{1.13\times10^4\,\text{MeV}}$

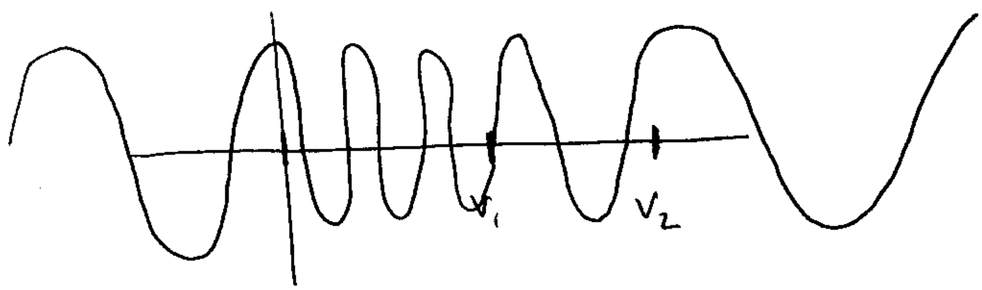
proton: $\boxed{6.15\,\text{MeV}}$

20] $-\dfrac{\partial E_n}{\partial L} = \dfrac{\hbar^2 \pi^2 n^2}{m L^3} = \boxed{1.2 \times 10^{-7}\,N}$

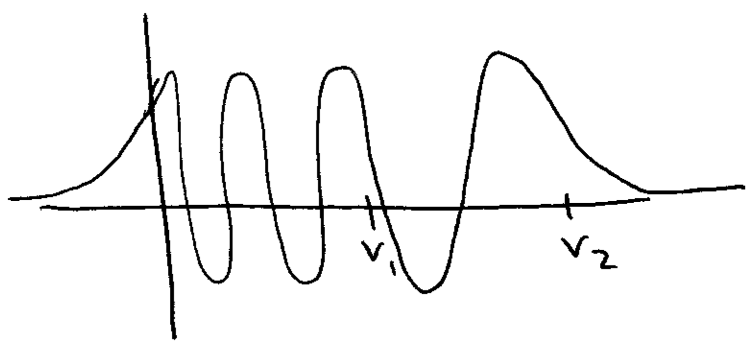
Weight of e^- $= mg = \boxed{8.9 \times 10^{-30}\,N}$ small!

25] For $E > V_2$, it looks like

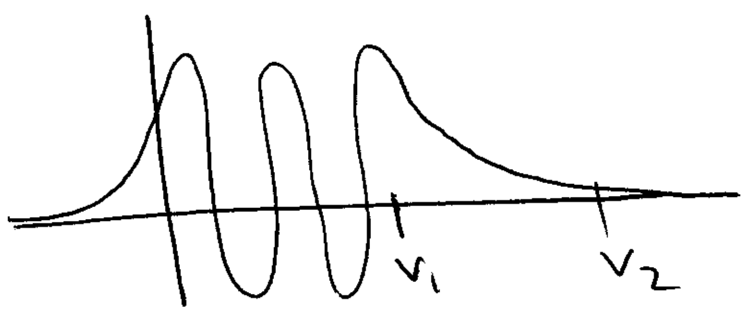
(big $E - U \Rightarrow$ big p $\Rightarrow$ big k $\Rightarrow$ smaller λ).



$V_1 < E < V_2$



$0 < E < V_1$



28a] $\langle x \rangle = \int_0^L \frac{2}{L} \sin^2\left(\frac{3\pi x}{L}\right) x \, dx$

Change vars to $u = \frac{3\pi x}{L} \Rightarrow \frac{2}{L}\left(\frac{L}{3\pi}\right)^2 \int_0^{3\pi} u \sin^2 u \, du$

$$= \frac{2}{9\pi^2 L}\left[\frac{u^2}{4} - \frac{u \sin 2u}{4} - \frac{\cos 2u}{8}\right|_0^{3\pi}$$

$$= \frac{2}{9\pi^2 L} \cdot \frac{9\pi^2}{4} = \boxed{\frac{L}{2}}$$

b] $\langle x^2 \rangle = \int_0^L \frac{2}{L} \sin^2\left(\frac{3\pi x}{L}\right) x^2 \, dx = \frac{2}{L}\left(\frac{L}{3\pi}\right)^3 \int_0^{3\pi} u^2 \sin^2 u \, du$

$$= \frac{2L^2}{27\cancel{9}\pi^3}\left[\frac{u^3}{6} - \left(\frac{u^2}{4} - \frac{1}{8}\right)\sin 2u - \frac{u \cos 2u}{4}\right|_0^{3\pi}$$

$$= \frac{2L^2}{27\cancel{9}\pi^3}\left[\frac{27\pi^3}{6} - \frac{3\pi}{4}\right]$$

$$= \boxed{L^2\left(\frac{1}{3} - \frac{1}{18\pi^2}\right)}$$

29] The probability of finding the particle in an interval of length dx is $\frac{dx}{L} = P(x)dx$.

$\therefore \boxed{P(x) = \frac{1}{L}}$

or (more rigorously) say $\frac{\text{time in } dx}{\text{time anywhere}} = \frac{dx/v}{L/v} = \frac{dx}{L} = P(x)dx$.

b] $\langle x \rangle = \int_0^L \frac{x}{L} dx = \frac{x^2}{2L}\Big|_0^L = \boxed{\frac{L}{2}}$

$\langle x^2 \rangle = \int_0^L \frac{x^2 dx}{L} = \frac{x^3}{3L}\Big|_0^L = \boxed{\frac{L^2}{3}}$

30] $\langle p^2 \rangle = \int \psi^* \left(-\hbar^2 \frac{\partial^2}{\partial x^2} \psi \right)$, but since $E\psi = \frac{-\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + V\psi$

$= \int \psi^* \left(2m(E-V)\psi \right) = \langle E - V \rangle 2m$

In the ∞ well, $V=0$, so $\langle p^2 \rangle = \langle 2mE \rangle$.

Ground state: $\langle p^2 \rangle = \frac{2m\hbar^2\pi^2}{2mL^2} = \boxed{\left(\frac{\hbar\pi}{L}\right)^2}$

31] $\langle p^2 \rangle = \left(\frac{\hbar\pi}{L}\right)^2$ from Prob. 30

$\langle p \rangle = 0$ by symmetry (particle is bouncing back & forth.)

$\Rightarrow \Delta p = \frac{\hbar\pi}{L}$.

$\langle x \rangle = \frac{L}{2}$, as in Prob. 28

$\langle x^2 \rangle = L^2\left(\frac{1}{3} - \frac{1}{2\pi^2}\right)$ (same method as Prob. 28)

$$\Rightarrow \boxed{\Delta x = \left[\frac{1}{12} - \frac{1}{2\pi^2}\right]^{1/2} L}$$

$$\Rightarrow \Delta x \Delta p = \hbar\pi\left(\frac{1}{12} - \frac{1}{2\pi^2}\right)^{1/2} = \boxed{\hbar\left[\frac{\pi^2}{12} - \frac{1}{2}\right]^{1/2} \approx 0.56\hbar}$$

Consistent w/ uncertainty principle!