4E : The Quantum Universe

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Vivek Sharma
modphys@hepmail.ucsd.edu
Just What is Waving in Matter Waves?

For waves in an ocean, it’s the water that “waves”
For sound waves, it’s the molecules in medium
For light it’s the \( \mathbf{E} \) & \( \mathbf{B} \) vectors that oscillate

Just What’s “waving” in matter waves?

- It’s the PROBABILITY OF FINDING THE PARTICLE that waves!
- Particle can be represented by a wave packet
  - At a certain location (x)
  - At a certain time (t)
  - Made by superposition of many sinusoidal waves of different amplitudes, wavelengths \( \lambda \) and frequency \( f \)
  - It’s a “pulse” of probability in spacetime
What Wave Does Not Describe a Particle

- What wave form can be associated with particle’s pilot wave?
- A traveling sinusoidal wave? \( y = A \cos (kx - \omega t + \Phi) \)
- Since de Broglie “pilot wave” represents particle, it must travel with same speed as particle ……(like me and my shadow)

Phase velocity \((v_p)\) of sinusoidal wave: \( v_p = \frac{\lambda}{f} \)

In Matter:

(a) \( \lambda = \frac{h}{p} = \frac{h}{\gamma mv} \)

(b) \( f = \frac{E}{h} = \frac{\gamma mc^2}{h} \)

\[ v_p = \frac{\lambda f}{p} = \frac{E}{\gamma mv} = \frac{\gamma mc^2}{v} > c! \]

Conflicts with Relativity \(\Rightarrow\) Unphysical

Single sinusoidal wave of infinite extent does not represent particle localized in space

Need “wave packets” localized Spatially \((x)\) and Temporally \((t)\)
Superposition of two sound waves of slightly different frequencies $f_1$ and $f_2$, $f_1 \approx f_2$

Pattern of beats is a series of wave packets

Beat frequency $f_{\text{beat}} = f_2 - f_1 = \Delta f$

$\Delta f = \text{range of frequencies that are superimposed to form the wave packet}$
Addition of 2 Waves with slightly different $\lambda$ and slightly different $\omega$

Resulting wave's "displacement" $y = y_1 + y_2$:

$$y = A\left[\cos(k_1x - w_1t) + \cos(k_2x - w_2t)\right]$$

Trigonometry: $\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$

$$\therefore y = 2A \left[\cos\left(\frac{k_2 - k_1}{2}x - \frac{w_2 - w_1}{2}t\right)\right]\cos\left(\frac{k_2 + k_1}{2}x - \frac{w_2 + w_1}{2}t\right)$$

since $k_2 \approx k_1 \approx k_{ave}$, $w_2 \approx w_1 \approx w_{ave}$, $\Delta k \ll k$, $\Delta w \ll w$

$$\therefore y = 2A \left[\cos\left(\frac{\Delta k}{2}x - \frac{\Delta w}{2}t\right)\right]\cos(kx - wt) \equiv y = A' \cos(kx - wt),$$

$A'$ oscillates in $x,t$; $A' = 2A \left(\frac{\Delta k}{2}x - \frac{\Delta w}{2}t\right) = \text{modulated amplitude}$

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**Phase Vel**

$$V_p = \frac{w_{ave}}{k_{ave}}$$

**Group Vel**

$$V_g = \frac{\Delta w}{\Delta k}$$

$V_g$ : Vel of envelope $= \frac{dw}{dk}$
Non-repeating wave packet can be created thru superposition of many waves of similar (but different) frequencies and wavelengths.

Waves to be added span the frequency range from $f_0 - \frac{1}{2}\Delta f$ to $f_0 + \frac{1}{2}\Delta f$.

The superposition of the many waves spanning a range of frequencies is a wave packet.

The waves are all in phase at this instant of time.
Wave Packet: Localization

Finite # of diff. Monochromatic waves always produce INFINITE sequence of repeating wave groups \(\Rightarrow\) can’t describe (localized) particle

To make localized wave packet, add “infinite” # of waves with
Well chosen Amplitude A, Wave number k and ang. frequency \(\omega\)

\[
\psi(x,t) = \int_{-\infty}^{\infty} A(k) e^{i(kx-\omega t)} dk
\]

\(A(k) = \text{Amplitude distribution } F_n\)
\(\Rightarrow\) diff waves of diff k
have different amplitudes \(A(k)\)
w = \(w(k)\), depends on type of wave, media

Group Velocity \(V_g = \frac{dw}{dk}\bigg|_{k=k_0}\)

\[\int\]
**Group Velocity, Phase Velocity and Dispersion**

In a Wave Packet: \( w = w(k) \)

Group Velocity \( V_g = \frac{dw}{dk} \) at \( k = k_0 \)

Since \( V_p = w k \) (def \( \Rightarrow \) \( w = k V_p \))

\[ V_g = \left. \frac{dw}{dk} \right|_{k=k_0} = V_p \left. \frac{dV_p}{dk} \right|_{k=k_0} \]

Material in which \( V_p \) varies with \( \lambda \) are said to be Dispersive.

Individual harmonic waves making a wave pulse travel at different \( V_p \) thus changing shape of pulse and become spread out.

In non-dispersive media, \( V_g = V_p \); Example: EM waves in vacuum.

Wave packet maintains its shape as it moves.

In dispersive media \( V_g \neq V_p \), depends on \( \frac{dV_p}{dk} \); shape changes with time.

Example: Water wave, EM waves in a medium.

1 ns laser pulse disperse by x30 after traveling 1km in optical fiber.
Example: Water Wave packet With $V_g = V_p/2$

Wave packet for which the group velocity $= 1/2$ phase velocity

The ↑, representing a point of constant phase for the dominant $\lambda$, travels with $V_p$

The ⊕ at center of group travels with group velocity ($V_g$)
A Dispersive Wave Packet Moving Along X Axis

The O indicates position of the classical particle. The Wave packet spreads out in x & y directions since $V_p$ of constituent waves depends on wavelength $\lambda$ of the wave.
Group Velocity $v_g$ of Matter Wave Packets

Consider an electron:

- Mass = $m$
- Velocity = $v$
- Momentum = $p$

Energy $E = hf = \gamma mc^2$;

\[ \omega = 2\pi f = \frac{2\pi}{\hbar} \gamma mc^2 \]

Wavelength $\lambda = \frac{\hbar}{p}$; $k = \frac{2\pi}{\lambda} \Rightarrow k = \frac{2\pi}{\hbar} \gamma mv$

Group Velocity:

\[ V_g = \frac{dw}{dk} = \frac{dw}{dv} \frac{dv}{dk} \]

\[ \frac{dw}{dv} = \frac{d}{dv} \left[ \frac{2\pi mc^2}{h \left[1 - \left(\frac{V}{c}\right)^2\right]^{3/2}} \right] = \frac{2\pi mv}{h \left[1 - \left(\frac{V}{c}\right)^2\right]^{3/2}} \]

\[ \frac{dk}{dv} = \frac{d}{dv} \left[ \frac{2\pi}{h \left[1 - \left(\frac{V}{c}\right)^2\right]^{1/2}} mv \right] = \frac{2\pi m}{h \left[1 - \left(\frac{V}{c}\right)^2\right]^{3/2}} \]

\[ V_g = \frac{dw}{dk} = \frac{dw}{dv} \frac{dv}{dk} = v \Rightarrow \text{Group velocity of electron Wave packet "pilot wave"}

is same as electron's physical velocity

But velocity of individual waves making up the wave packet

\[ V_p = \frac{w}{k} = \frac{c^2}{v} > c! \] (not physical)
Wave Packets & Uncertainty Principles

- Distance $\Delta X$ between adjacent minima = $(X_2)_\text{node} - (X_1)_\text{node}$
- Define $X_1=0$ then phase diff from $X_1 \Rightarrow X_2 = \pi$ (similarly for $t_1 \Rightarrow t_2$)

Node at $y = 0 = 2A \cos (\frac{\Delta w}{2} - \frac{\Delta k}{2}x)$, Examine $x$ or $t$ behavior

$\Rightarrow \text{ in } x: \Delta k. \Delta x = \pi \Rightarrow \text{ Need to combine many waves of diff. } k \text{ to make small } \Delta x \text{ pulse}$

$\Delta x = \frac{\pi}{\Delta k}$, for small $\Delta x \rightarrow 0 \Rightarrow \Delta k \rightarrow \infty$ & Vice Verca

$\text{and in } t: \Delta w. \Delta t = \pi \Rightarrow \text{ Need to combine many waves of diff } \omega \text{ to make small } \Delta t \text{ pulse}$

$\Delta t = \frac{\pi}{\Delta \omega}$, for small $\Delta t \rightarrow 0 \Rightarrow \Delta \omega \rightarrow \infty$ & Vice Verca
• Short duration pulses are used to transmit digital info
  – Over phone line as brief tone pulses
  – Over satellite link as brief radio pulses
  – Over optical fiber as brief laser light pulses
• Regardless of type of wave or medium, any wave pulse must obey the fundamental relation
  \[ \Delta \omega \Delta t \cong \pi \]
• Range of frequencies that can be transmitted are called bandwidth of the medium
• Shortest possible pulse that can be transmitted thru a medium is \[ \Delta t_{\text{min}} \cong \frac{\pi}{\Delta \omega} \]
• Higher bandwidths transmits shorter pulses & allows high data rate
Wave Packets & The Uncertainty Principles of Subatomic Physics

in space x: \( \Delta k \cdot \Delta x = \pi \)

\[ \Rightarrow \text{since} \quad k = \frac{2\pi}{\lambda}, \quad p = \frac{\hbar}{\lambda} \]

\[ \Rightarrow \quad \Delta p \cdot \Delta x = \hbar / 2 \]

usually one writes \( \Delta p \cdot \Delta x \geq \hbar / 2 \) approximate relation

In time t : \( \Delta \omega \cdot \Delta t = \pi \)

\[ \Rightarrow \text{since} \quad \omega = 2\pi f, \quad E = hf \]

\[ \Rightarrow \quad \Delta E \cdot \Delta t = \hbar / 2 \]

usually one writes \( \Delta E \cdot \Delta t \geq \hbar / 2 \) approximate relation

What do these inequalities mean physically?
Know the Error of Thy Ways: Measurement Error $\Delta$

- Measurements are made by observing something: length, time, momentum, energy.
- All measurements have some (limited) precision...no matter the instrument used.
- Examples:
  - How long is a desk? $L = (5 \pm 0.1) \text{ m} = L \pm \Delta L$ (depends on ruler used).
  - How long was this lecture? $T = (50 \pm 1)\text{ minutes} = T \pm \Delta T$ (depends on the accuracy of your watch).
  - How much does Prof. Sharma weigh? $M = (1000 \pm 700) \text{ kg} = m \pm \Delta m$.
    - Is this a correct measure of my weight?
      - Correct (because of large error reported) but imprecise.
      - My correct weight is covered by the (large) error in observation.

Length Measure

Voltage (or time) Measure
Measurement Error: \( x \pm \Delta x \)

- Measurement errors are unavoidable since the measurement procedure is an experimental one.
- True value of an measurable quantity is an abstract concept.
- In a set of repeated measurements with random errors, the distribution of measurements resembles a Gaussian distribution characterized by the parameter \( \sigma \) or \( \Delta \) characterizing the width of the distribution.

The Gauss, or Normal, Distribution

\[
G_{X,\sigma}(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x - \bar{x})^2/2\sigma^2}.
\]
**Measurement Error**: $x \pm \Delta x$
Interpreting Measurements with random Error: $\Delta$

**Figure 5.12.** The shaded area between $X \pm t\sigma$ is the probability of a measurement within $t$ standard deviations of $X$. 

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<th>0</th>
<th>0.25</th>
<th>0.5</th>
<th>0.75</th>
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<td>98.8</td>
<td>99.7</td>
<td>99.95</td>
<td>99.99</td>
</tr>
</tbody>
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Where in the World is Carmen San Diego?

Carmen San Diego hidden inside a big box of length $L$

Suppose you can’t see thru the (blue) box, what is you best estimate of her location inside box (she could be anywhere inside the box)

Your best unbiased measure would be $x = L/2 \pm L/2$

There is no perfect measurement, there are always measurement error
Wave Packets & Matter Waves

• What is the Wave Length of this wave packet?
  • made of waves with $\lambda - \Delta \lambda < \lambda < \lambda + \Delta \lambda$
  • De Broglie wavelength $\lambda = h/p$
    • $\Rightarrow$ Momentum Uncertainty: $p - \Delta p < p < p + \Delta p$
• Similarly for frequency $\omega$ or $f$
  • made of waves with $\omega - \Delta \omega < \omega < \omega + \Delta \omega$
  • Planck’s condition $E = hf = \hbar \omega / 2$
    • $\Rightarrow$ Energy Uncertainty: $E - \Delta E < E < E + \Delta E$
Back to Heisenberg’s Uncertainty Principle

• $\Delta x \cdot \Delta p \geq \hbar/4\pi \Rightarrow$ If the measurement of the position of a particle is made with a precision $\Delta x$ and a SIMULTANEOUS measurement of its momentum $p_x$ in the X direction, then the product of the two uncertainties (measurement errors) can never be smaller than $\geq \hbar/4\pi$ irrespective of how precise the measurement tools

• $\Delta E \cdot \Delta t \geq \hbar/4\pi \Rightarrow$ If the measurement of the energy $E$ of a particle is made with a precision $\Delta E$ and it took time $\Delta t$ to make that measurement, then the product of the two uncertainties (measurement errors) can never be smaller than $\geq \hbar/4\pi$ irrespective of how precise the measurement tools

These rules arise from the way we constructed the wave packets describing Matter “pilot” waves

Perhaps these rules are bogus, can we verify this with some physical picture??
Are You Experienced?

• What you experience is what you observe
• What you observe is what you measure
• No measurement is perfect, they all have measurement error: question is of the degree
  – Small or large $\Delta$

• Uncertainty Principle and Breaking of Conservation Rules
  – Energy Conservation
  – Momentum Conservation
Observations of particle motion by means of scattered illumination. When the incident wavelength is reduced to accommodate the size of the particle, the momentum transferred by the photon becomes large enough to disturb the observed motion.

Visible light illuminating a macroscopic object.

Act of observation disturbs the observed system.

X ray illuminating an atomic electron.
Act of Observation Tells All
Compton Scattering: Shining light to observe electron

Photon scattering off an electron, Seeing→ the photon enters my eye

\[ \lambda = \frac{h}{p} = \frac{hc}{E} = \frac{c}{f} \]

The act of Observation DISTURBS the object being watched, here the electron moves away from where it was originally
Act of Watching: A Thought Experiment

Before collision
- Incident photon
- Electron

After collision
- Scattered photon
- Recoiling electron

Observed Diffraction pattern
Photons that go through are restricted to this region of lens

Lens

Scattered photon $\hat{p} = \frac{h}{\lambda}$

Electron $e^-$ initially at rest
$\Delta x$

Incident photon $p_0 = \frac{h}{\lambda_0}$
Diffraction By a Circular Aperture (Lens)

See Resnick, Halliday Walker 6th Ed (on S.Reserve), Ch 37, pages 898-900

Diffracted image of a point source of light thru a lens (circular aperture of size \( d \))

First minimum of diffraction pattern is located by

\[
\sin \theta = 1.22 \frac{\lambda}{d}
\]

See previous picture for definitions of \( \theta, \lambda, d \)

Fig. 37-9 The diffraction pattern of a circular aperture. Note the central maximum and the circular secondary maxima. The figure has been overexposed to bring out these secondary maxima, which are much less intense than the central maximum.
Resolving Power of Light Thru a Lens

Image of 2 separate point sources formed by a converging lens of diameter $d$, ability to resolve them depends on $\lambda$ & $d$ because of the Inherent diffraction in image formation.

\[ \Delta x \approx \frac{\lambda}{2\sin \theta} \]

$\theta$ depends on lens radius $d$. 

Not resolved, Barely resolved, Resolved.
Putting it all together: Act of Observing an Electron

- Incident light \((p, \lambda)\) scatters off electron
- To be collected by lens \(\rightarrow \gamma\) must scatter thru angle \(\alpha\)
  - \(-9 \leq \alpha \leq 9\)
- Due to Compton scatter, electron picks up momentum
  - \(P_x, P_y\)
- After passing thru lens, photon diffracts, lands somewhere on screen, image (of electron) is fuzzy
- How fuzzy? Optics says shortest distance between two resolvable points is:
  \[
  \Delta x = \frac{\lambda}{2 \sin \theta}
  \]
- Larger the lens radius, larger the \(\theta\) \(\Rightarrow\) better resolution

\[\Rightarrow \Delta p \cdot \Delta x \approx \left(\frac{2h \sin \theta}{\lambda}\right) \left(\frac{\lambda}{2 \sin \theta}\right) = \hbar\]
Aftermath of Uncertainty Principle

- Deterministic (Newtonian) physics topples over
  - Newton’s laws told you all you needed to know about the trajectory of a particle
    - Apply a force, watch the particle go!
      - Know everything! \( X, v, p, F, a \)
      - Can predict exact trajectory of particle if you had perfect device
  - No so in the subatomic world!
    - Of small momenta, forces, energies
    - Can’t predict anything exactly
      - Can only predict probabilities
        - There is so much chance that the particle landed here or there
        - Cant be sure!...cognizant of the errors of thy observations
All Measurements Have Associated Errors

• If your measuring apparatus has an intrinsic inaccuracy (error) of amount $\Delta p$

• Then results of measurement of momentum $p$ of an object at rest can easily yield a range of values accommodated by the measurement imprecision:
  \[-\Delta p \leq p \leq \Delta p\] : you will measure any of these values for the momentum of the particle

• Similarly for all measurable quantities like $x$, $t$, Energy!
Matter Diffraction & Uncertainty Principle

Incident Electron beam In Y direction

slit size: a

Momentum measurement beyond slit show particle not moving exactly in Y direction, develops a X component Of motion\[-\Delta p_x \leq p_x \leq \Delta p_x\] \textbf{with}
\[\Delta p_x = \frac{h}{2\pi a}\]

X component $P_X$ of momentum
Object of mass M at rest between two walls originally at infinity.

What happens to our perception of George’s momentum as the walls are brought in?

On average, measure $<p> = 0$

but there are quite large fluctuations!

Width of Distribution $= \Delta P$

$$\Delta P = \sqrt{(P^2)_{ave} - (P_{ave})^2}; \quad \Delta P \sim \frac{\hbar}{L}$$