4E : The Quantum Universe

Lecture 9, April 13

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Just What is Waving in Matter Waves ?

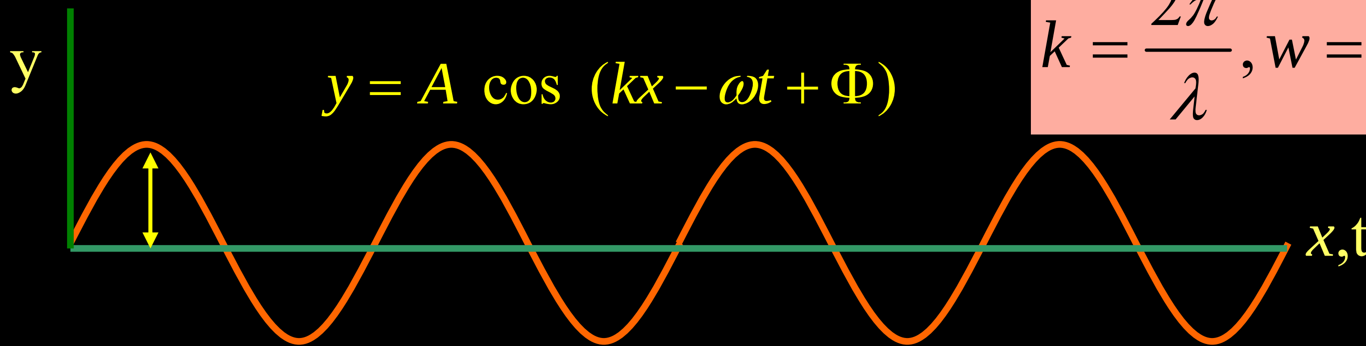
For waves in an ocean, it's the water that "waves"
For sound waves, it's the molecules in medium
For light it's the **E** & **B** vectors that oscillate

Just What's "**waving**" in matter waves ?

- **It's the PROBABLILITY OF FINDING THE PARTICLE that waves !**
- **Particle can be represented by a wave packet**
  - At a certain location (x)
  - At a certain time (t)
  - **Made by superposition of many sinusoidal waves of different amplitudes, wavelengths λ and frequency f**
  - **It's a "pulse" of probability in spacetime**

What Wave Does Not Describe a Particle

$$k = \frac{2\pi}{\lambda}, w = 2\pi f$$



- What wave form can be associated with particle's pilot wave?
- A traveling sinusoidal wave? $y = A \cos(kx - \omega t + \Phi)$
- Since de Broglie "pilot wave" represents particle, it must travel with same speed as particle ……(like me and my shadow)

Phase velocity (v_p) of sinusoidal wave: $v_p = \lambda f$

In Matter:

$$(a)\quad \lambda = \frac{h}{p} = \frac{h}{\gamma m v}$$

$$(b)\quad f = \frac{E}{h} = \frac{\gamma m c^2}{h}$$

$$\Rightarrow v_p = \lambda f = \frac{E}{p} = \frac{\gamma m c^2}{\gamma m v} = \frac{c^2}{v} > c!$$

Conflicts with Relativity → Unphysical

Single sinusoidal wave of infinite extent does not represent particle localized in space

Need "wave packets" localized Spatially (x) and Temporally (t)

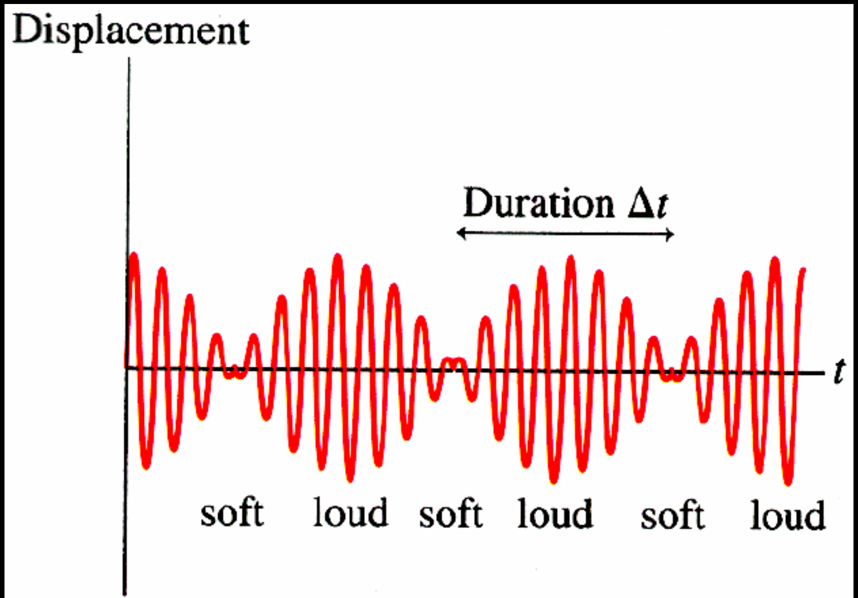
How To Make Wave Packets : Just Beat it !

Superposition of two sound waves of slightly different frequencies f_1 and f_2 , $f_1 \cong f_2$

Pattern of beats is a series of wave packets

Beat frequency $f_{beat} = f_2 - f_1 = \Delta f$

Δf = range of frequencies that are superimposed to form the wave packet



Addition of 2 Waves with slightly different λ and slightly different ω

Resulting wave's "displacement" $y = y_1 + y_2$:

$$y = A\left[\cos(k_1 x - w_1 t) + \cos(k_2 x - w_2 t)\right]$$

Trignometry : $\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$

$$\therefore\ y = 2A\left[\left(\cos\left(\frac{k_2 - k_1}{2}x - \frac{w_2 - w_1}{2}t\right)\right)\left(\cos\left(\frac{k_2 + k_1}{2}x - \frac{w_2 + w_1}{2}t\right)\right)\right]$$

since $k_2 \cong k_1 \cong k_{ave}$, $w_2 \cong w_1 \cong w_{ave}$, $\Delta k \ll k, \Delta w \ll w$

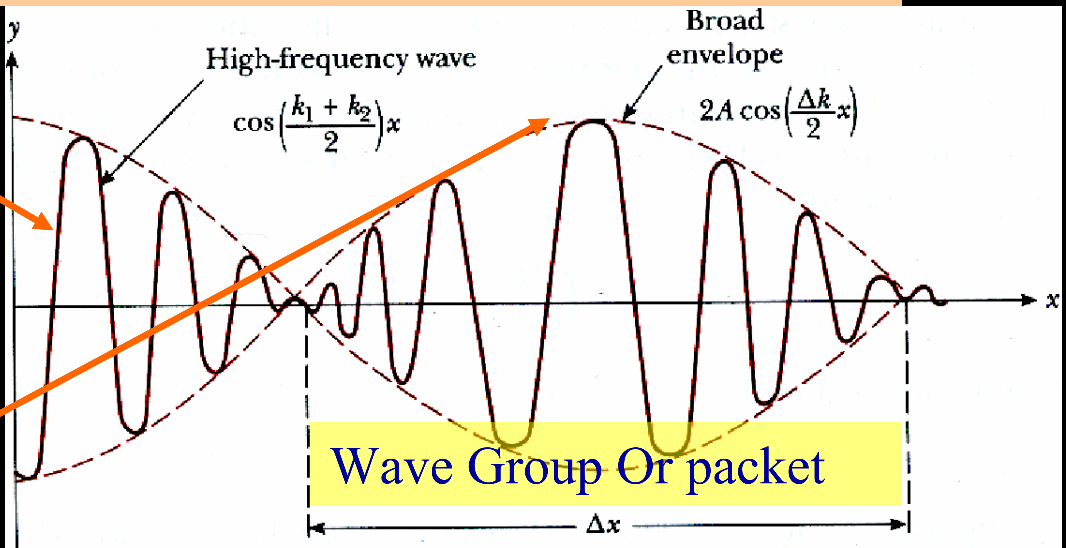
$$\therefore\ \boxed{y = 2A\left[\left(\cos\left(\frac{\Delta k}{2}x - \frac{\Delta w}{2}t\right)\right)\cos(kx - wt)\right]} \equiv\ y = A'\cos(kx - wt),$$

A' oscillates in x,t ; $A' = 2A\left(\cos\left(\frac{\Delta k}{2}x - \frac{\Delta w}{2}t\right)\right)$ = modulated amplitude

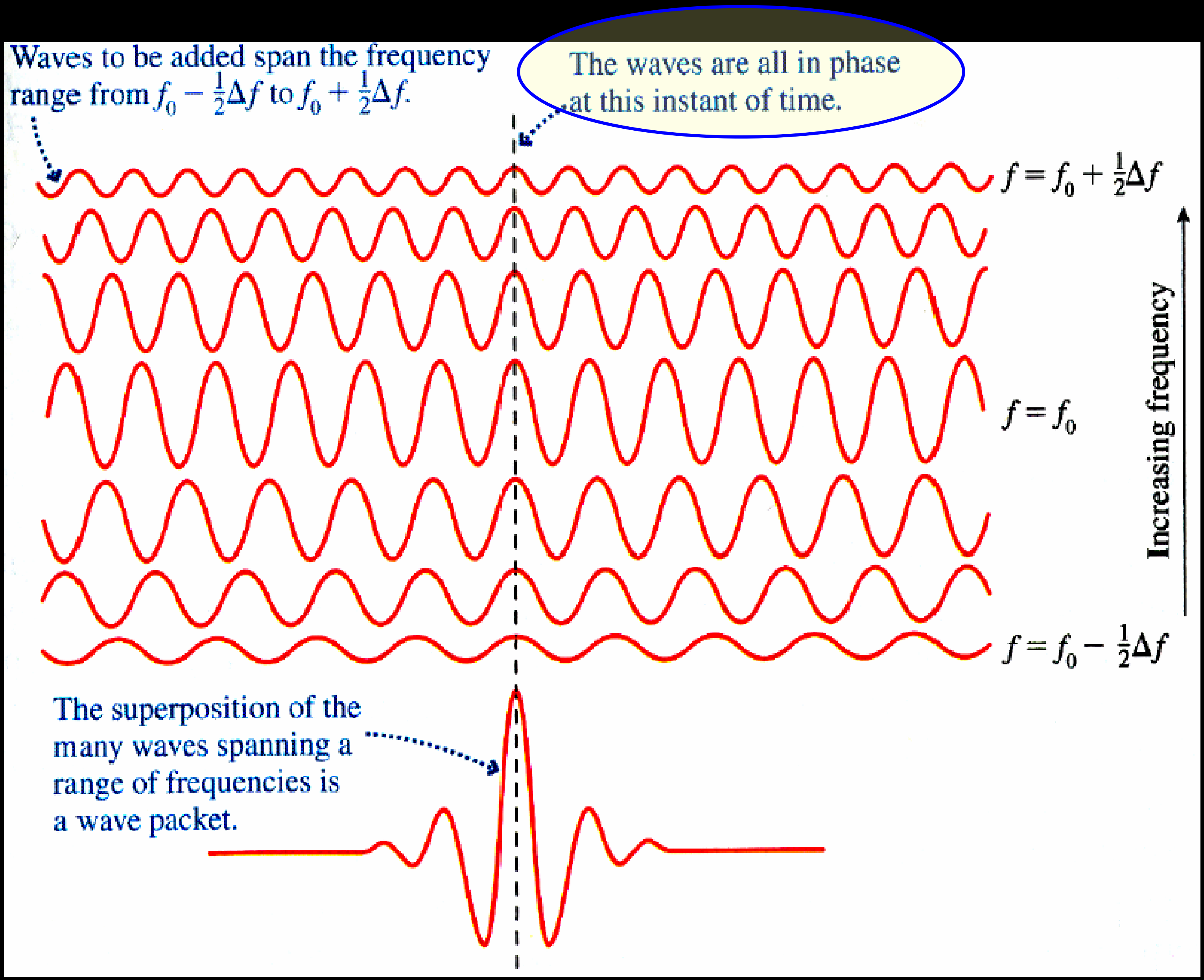
Phase Vel $\quad V_p = \frac{w_{ave}}{k_{ave}}$

Group Vel $\quad V_g = \frac{\Delta w}{\Delta k}$

V_g : Vel of envelope$=\frac{dw}{dk}$



Non-repeating wave packet can be created thru superposition Of many waves of similar (but different) frequencies and wavelengths



Wave Packet : Localization

Finite # of diff. Monochromatic waves always produce INFINTE sequence of repeating wave groups → can't describe (localized) particle

To make localized wave packet, add " infinite" # of waves with Well chosen Amplitude A, Wave number k and ang. f requency ω

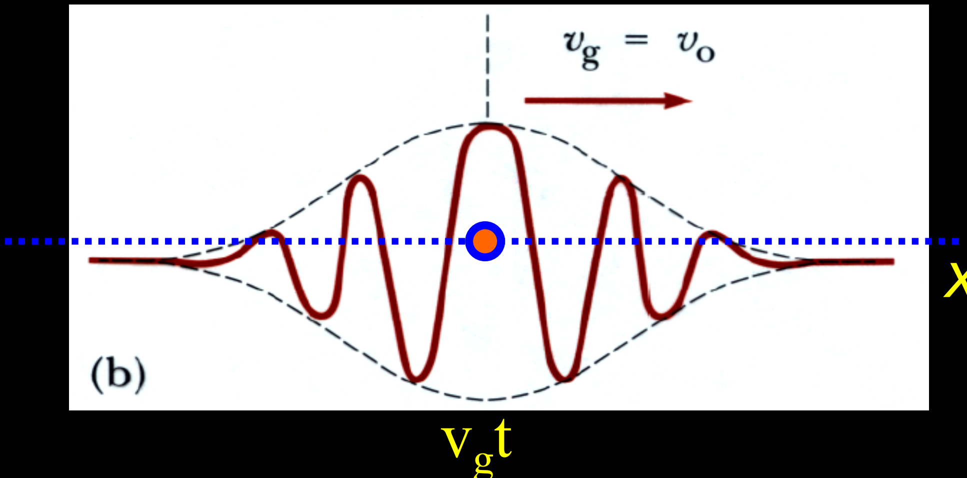
$$\psi(x,t) = \int_{-\infty}^{\infty} A(k)\, e^{i(kx-wt)} dk$$

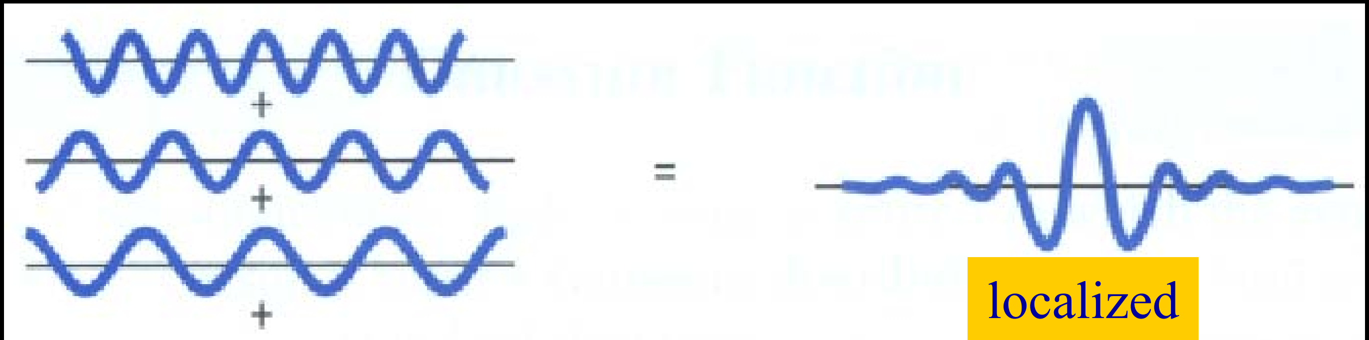
$A(k)$ = Amplitude distribution Fn

$\Rightarrow$ diff waves of diff k have different amplitudes A(k)

w = w(k), depends on type of wave, media

Group Velocity $V_g = \left.\dfrac{dw}{dk}\right|_{k=k_0}$





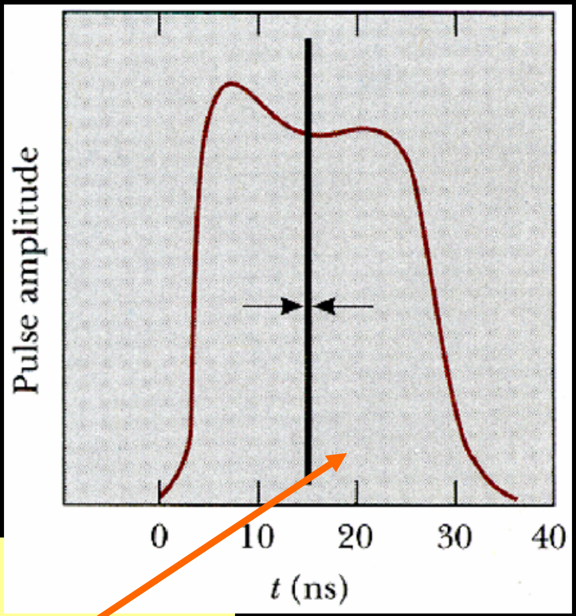
Group Velocity, Phase Velocity and Dispersion

In a Wave Packet: $w = w(k)$

Group Velocity $V_g = \left.\frac{dw}{dk}\right|_{k=k_0}$

Since $V_p = wk \quad (def) \Rightarrow w = kV_p$

$$\therefore \boxed{V_g = \left.\frac{dw}{dk}\right| = V_p\Big|_{k=k_0} + k\left.\frac{dV_p}{dk}\right|_{k=k_0}}$$



usually $V_p = V_p(\lambda)$

Material in which V_p varies with λ are said to be Dispersive

Individual harmonic waves making a wave pulse travel at different V_p thus changing shape of pulse and become spread out

In non-dispersive media, $V_g = V_p$; Example: EM waves in vaccum

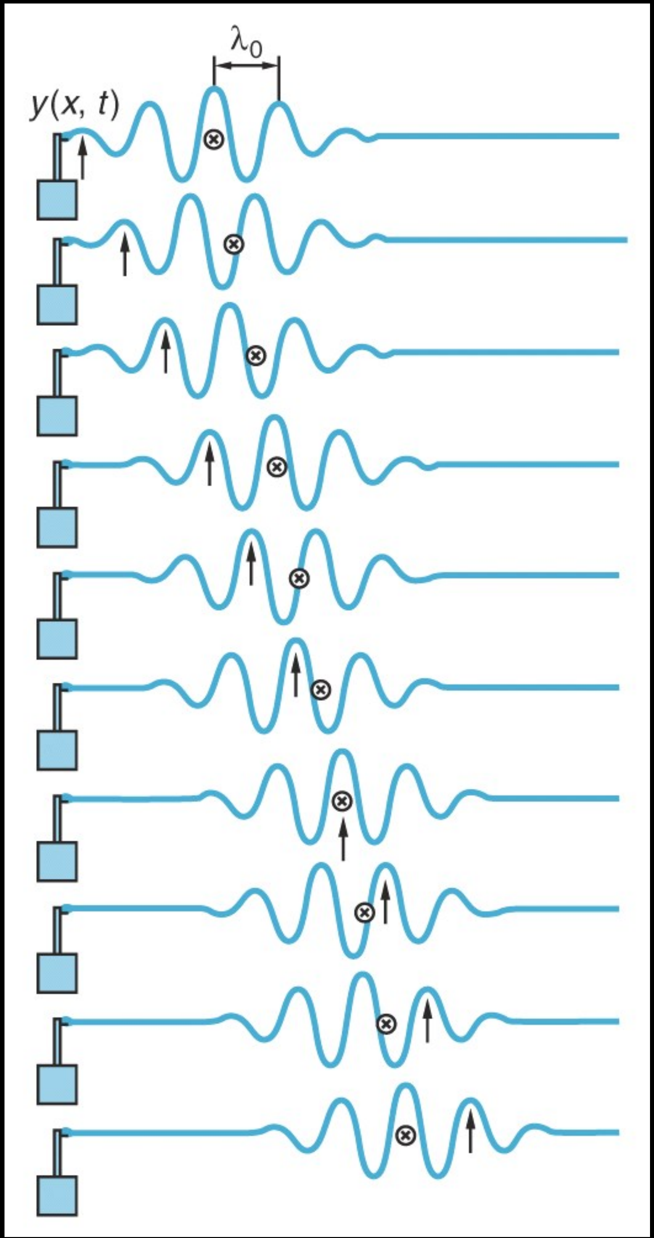
Wave packet maintains its shape as it moves.

In dispersive media $V_g \neq V_p$, depends on $\frac{dV_p}{dk}$; shape changes with time

Example: Water wave, EM waves in a medium

1 ns laser pulse disperse by x30 after traveling 1km in optical fiber

Example: Water Wave packet With $V_g=V_p/2$

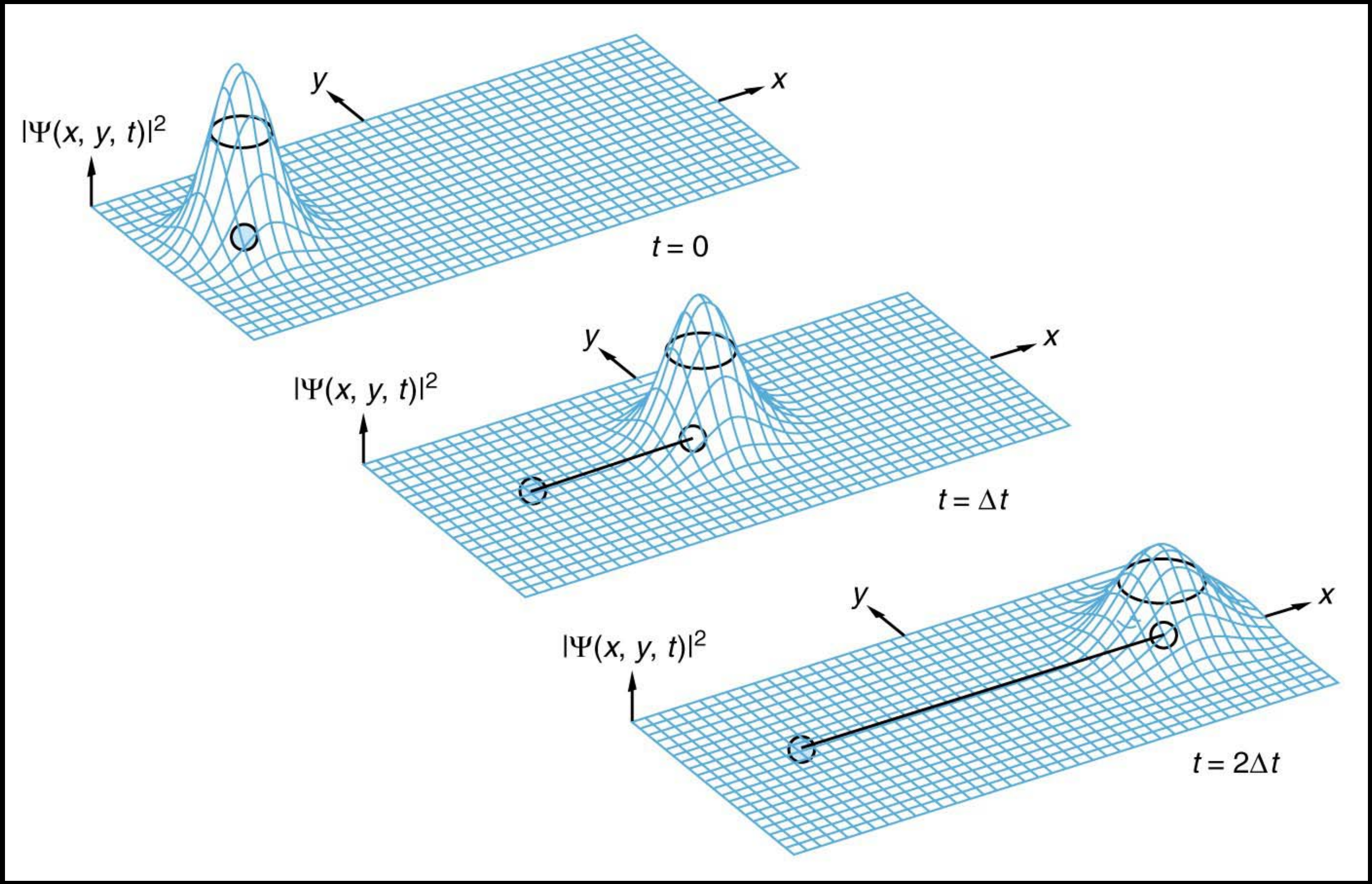


Wave packet for which the group velocity=1/2 phase velocity

The ↑, representing a point of constant phase for the dominant λ, travels with V_p

The ⊕ at center of group travels with group velocity (V_g)

A Dispersive Wave Packet Moving Along X Axis



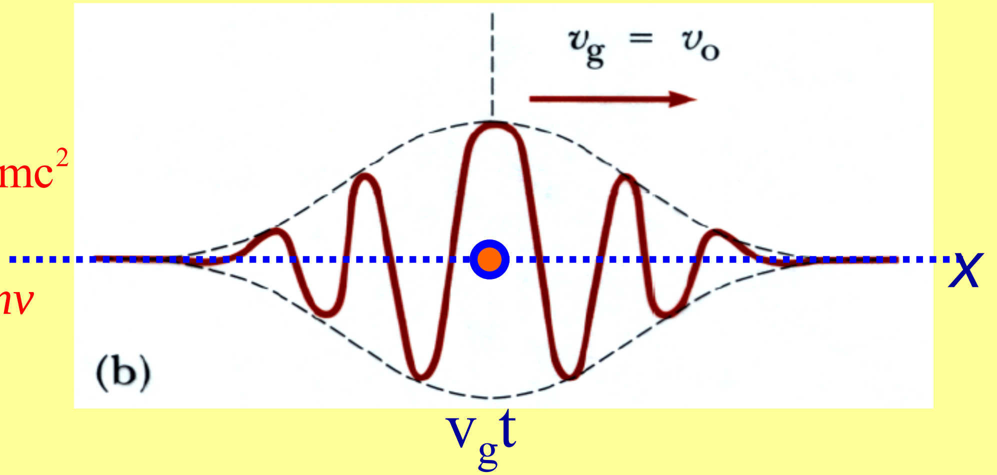
The O indicates position of the classical particle. The Wave packet spreads out in x & y directions since V_p of constituent waves depends on wavelength λ of the wave

Group Velocity v_g of Matter Wave Packets

Consider An Electron:

mass = m velocity = v, momentum = p

Energy $E = hf = \gamma mc^2$; $\omega = 2\pi f = \dfrac{2\pi}{h}\gamma mc^2$



Wavelength $\lambda = \dfrac{h}{p}$; $k = \dfrac{2\pi}{\lambda} \Rightarrow k = \dfrac{2\pi}{h}\gamma mv$

Group Velocity : $V_g = \dfrac{dw}{dk} = \dfrac{dw/dv}{dk/dv}$

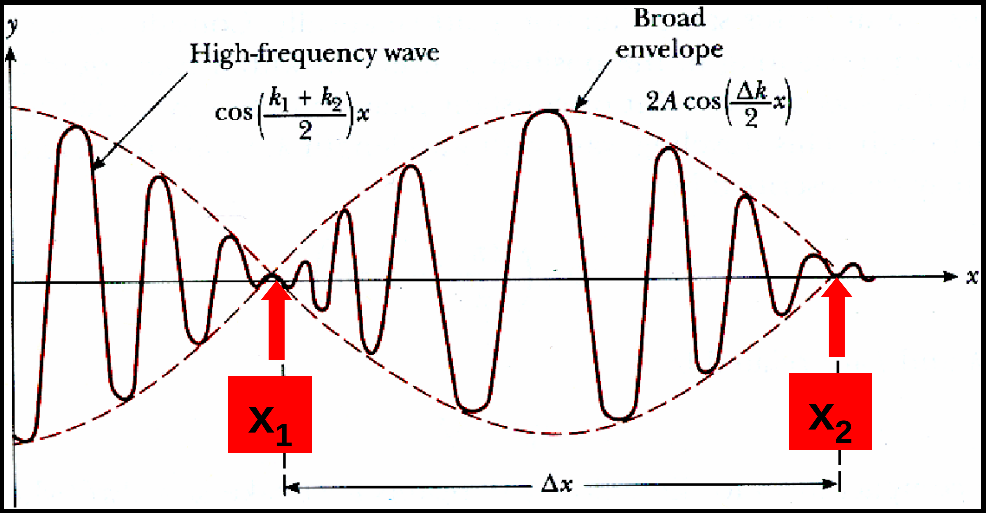
$$\frac{dw}{dv} = \frac{d}{dv}\left[\frac{\frac{2\pi}{h}mc^2}{[1-(\frac{v}{c})^2]^{1/2}}\right] = \frac{2\pi mv}{h[1-(\frac{v}{c})^2]^{3/2}} \quad \& \quad \frac{dk}{dv} = \frac{d}{dv}\left[\frac{2\pi}{h[1-(\frac{v}{c})^2]^{1/2}}mv\right] = \frac{2\pi m}{h[1-(\frac{v}{c})^2]^{3/2}}$$

$V_g = \dfrac{dw}{dk} = \dfrac{dw/dv}{dk/dv} = v \Rightarrow$ Group velocity of electron Wave packet "pilot wave"

is same as electron's physical velocity

But velocity of individual waves making up the wave packet $V_p = \dfrac{w}{k} = \dfrac{c^2}{v} > c$! (not physical)

Wave Packets & Uncertainty Principles



$$y = 2A\left[\left(\cos\left(\frac{\Delta k}{2}x - \frac{\Delta w}{2}t\right)\right)\cos(kx - wt)\right]$$

What can we learn from this simple model ?

- Distance ΔX between adjacent minima = $(X_2)_{node} - (X_1)_{node}$
- Define X_1=0 then phase diff from $X_1 \rightarrow X_2 = \pi$ (similarly for $t_1 \rightarrow t_2$)

Node at $y = 0 = 2A\cos\left(\frac{\Delta w}{2}t - \frac{\Delta k}{2}x\right)$, Examine x or t behavior

$\Rightarrow$ in x: $\boxed{\Delta k.\Delta x = \pi}$ $\Rightarrow$ Need to combine many waves of diff. k to make small Δx pulse

$$\Delta x = \frac{\pi}{\Delta k}, \text{ for small } \Delta x \to 0 \Rightarrow \Delta k \to \infty \text{ \& Vice Verca}$$

and In t : $\boxed{\Delta w.\Delta t = \pi}$ $\Rightarrow$ Need to combine many waves of diff ω to make small Δt pulse

$$\Delta t = \frac{\pi}{\Delta \omega}, \text{ for small } \Delta t \to 0 \Rightarrow \Delta \omega \to \infty \text{ \& Vice Verca}$$

Signal Transmission and Bandwidth Theory

- Short duration pulses are used to transmit digital info
  - Over phone line as brief tone pulses
  - Over satellite link as brief radio pulses
  - Over optical fiber as brief laser light pulses
- Ragardless of type of wave or medium, any wave pulse must obey the fundamental relation
  - $\Delta\omega\Delta t \cong \pi$
- Range of frequencies that can be transmitted are called bandwidth of the medium
- Shortest possible pulse that can be transmitted thru a medium is $\Delta t_{min} \cong \pi/\Delta\omega$
- Higher bandwidths transmits shorter pulses & allows high data rate

in space x: $\boxed{\Delta k.\Delta x = \pi}$ $\Rightarrow$ since $\mathrm{k} = \frac{2\pi}{\lambda}$, $\mathrm{p} = \frac{\mathrm{h}}{\lambda}$

$\Rightarrow$ $\boxed{\Delta p.\Delta x = h/2}$

usually one writes $\boxed{\Delta p.\Delta x \geq \hbar/2}$ approximate relation

In time t : $\boxed{\Delta w.\Delta t = \pi}$ $\Rightarrow$ since $\omega = 2\pi f, E = hf$

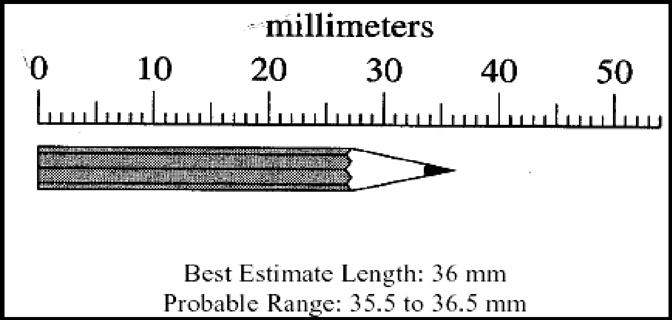
$\Rightarrow$ $\boxed{\Delta E.\Delta t = h/2}$

usually one writes $\boxed{\Delta E.\Delta t \geq \hbar/2}$ approximate relation

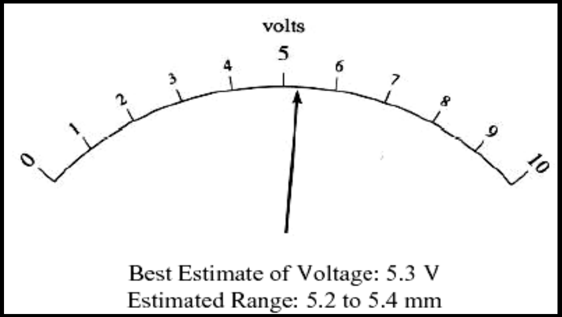
What do these inequalities mean physically?

Know the Error of Thy Ways: Measurement Error → Δ

- Measurements are made by observing something : length, time, momentum, energy
- All measurements have some (limited) precision….<u>no matter</u> the instrument used
- Examples:
  - How long is a desk ? $L = (5 \pm 0.1)$ m $= L \pm \Delta L$ (depends on ruler used)
  - How long was this lecture ? $T = (50 \pm 1)$minutes $= T \pm \Delta T$ (depends on the accuracy of your watch)
  - How much does Prof. Sharma weigh ? $M = (1000 \pm 700)$ kg $= m \pm \Delta m$
    - Is this a correct measure of my weight ?
      - Correct (because of large error reported) but imprecise
      - My correct weight is covered by the (large) error in observation



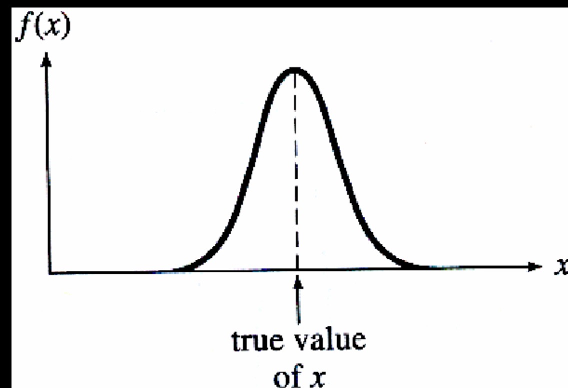
Length Measure



Voltage (or time) Measure

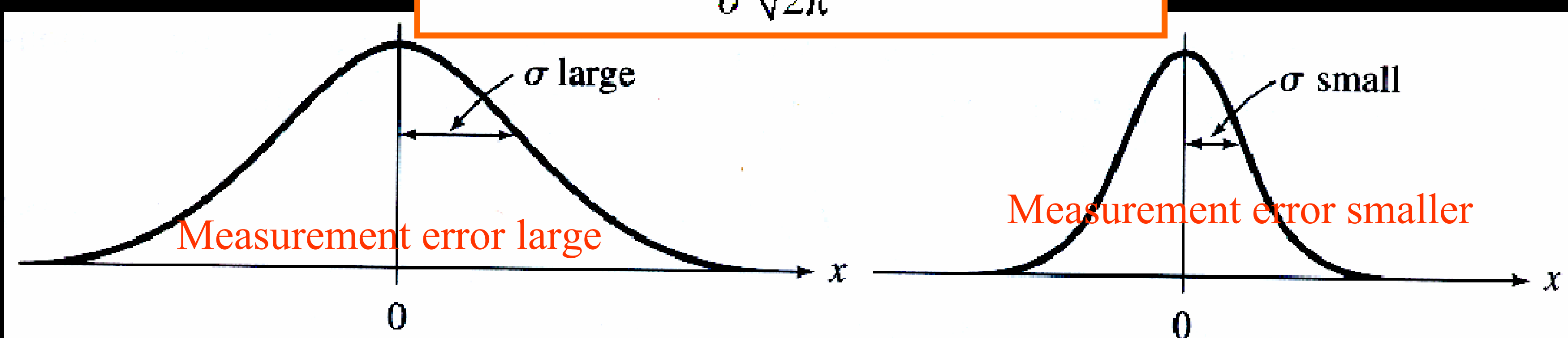
Measurement Error : x ± Δx

- Measurement errors are unavoidable since the measurement procedure is an experimental one
- True value of an measurable quantity is an abstract concept
- In a set of repeated <u>measurements with random errors</u>, the distribution of measurements resembles a Gaussian distribution characterized by the parameter σ or Δ characterizing the width of the distribution

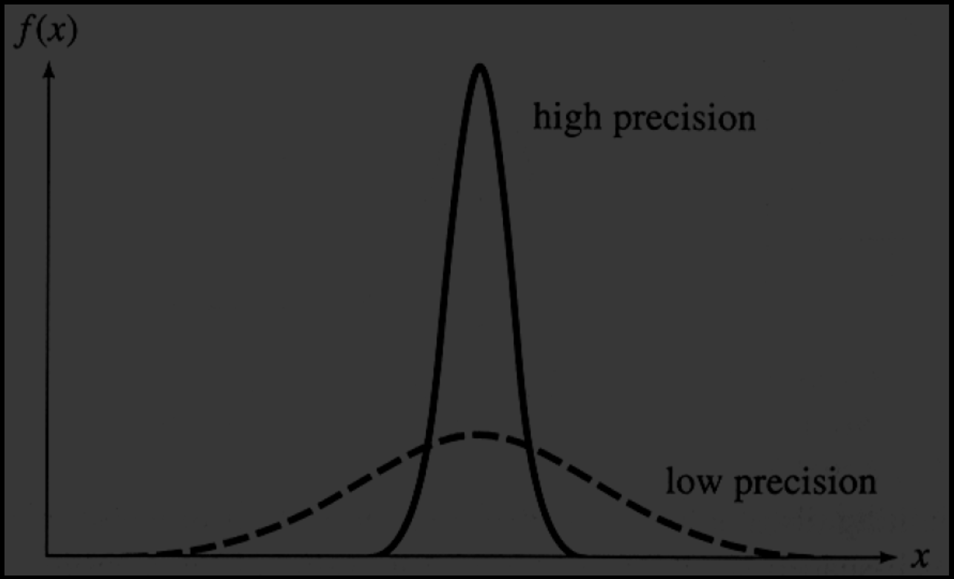


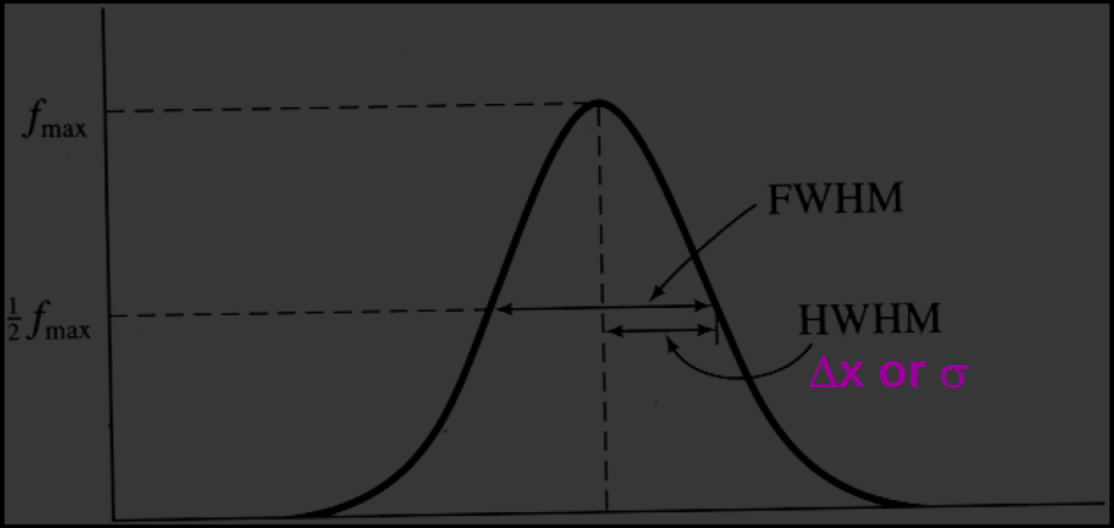
The Gauss, or Normal, Distribution

$$G_{X,\sigma}(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{(x-X)^2/2\sigma^2}.$$



Measurement Error : $x \pm \Delta x$





Interpreting Measurements with random Error : Δ

Will use Δ = σ interchangeably

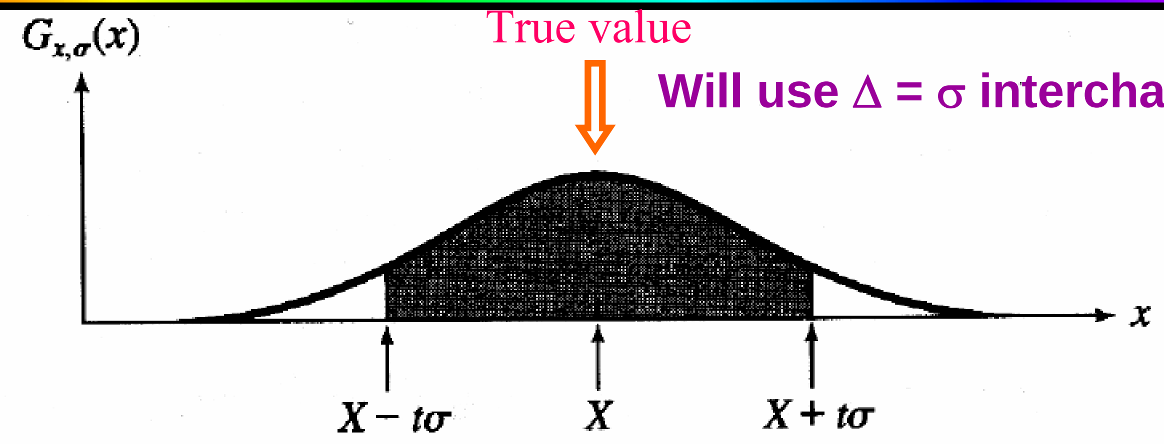
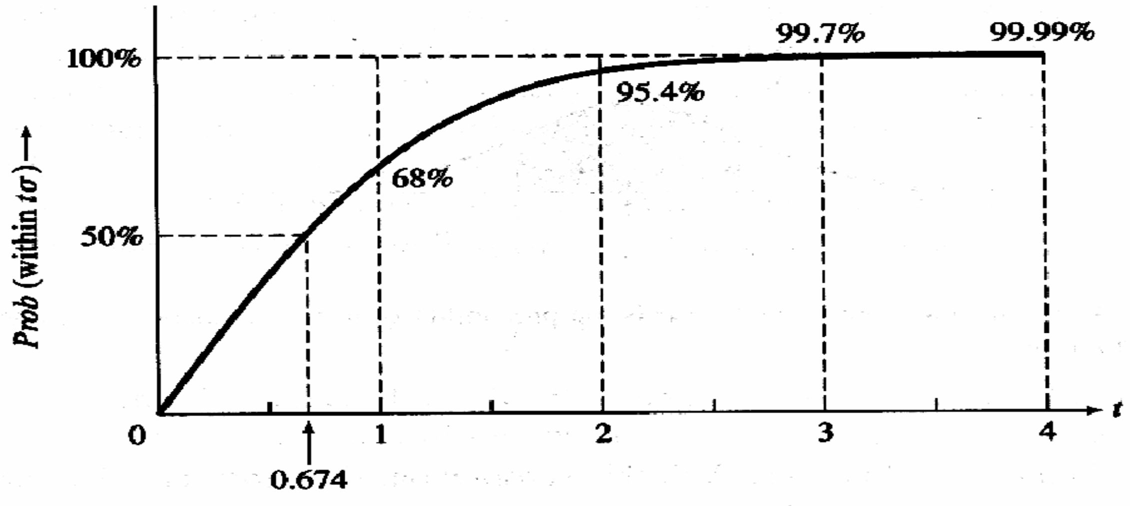


Figure 5.12. The shaded area between $X \pm t\sigma$ is the probability of a measurement within t standard deviations of X.



t	0	0.25	0.5	0.75	1.0	1.25	1.5	1.75	2.0	2.5	3.0	3.5	4.0
Prob (%)	0	20	38	55	68	79	87	92	95.4	98.8	99.7	99.95	99.99

Where in the World is Carmen San Diego?

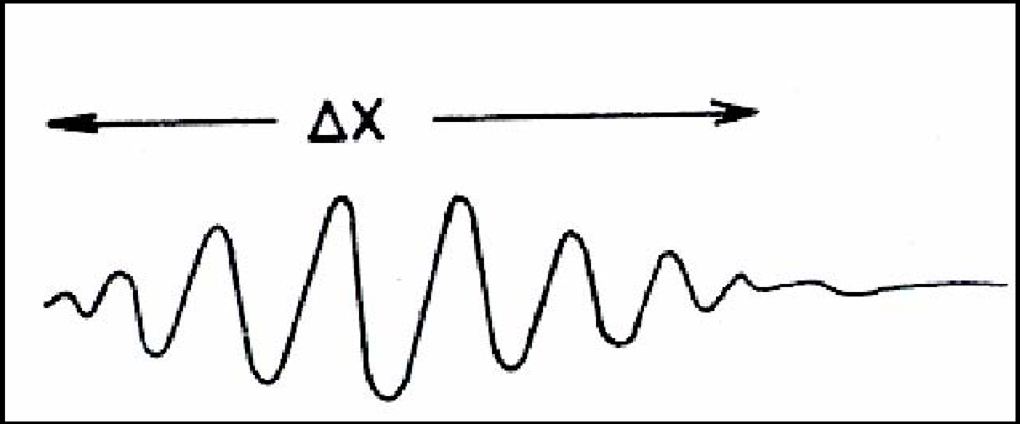
Carmen San Diego hidden inside a big box of length L

Suppose you can't see thru the (blue) box, what is you best estimate of her location inside box (she could be anywhere inside the box)

Your best unbiased measure would be $x = L/2 \pm L/2$

There is no perfect measurement, there are always measurement error

Wave Packets & Matter Waves



- What is the Wave Length of this wave packet?
  - made of waves with $\lambda - \Delta\lambda < \lambda < \lambda + \Delta\lambda$
  - De Broglie wavelength $\lambda = h/p$
    - → Momentum Uncertainty: $p - \Delta p < p < p + \Delta p$
- Similarly for frequency ω or f
  - made of waves with $\omega - \Delta\omega < \omega < \omega + \Delta\omega$

Planck's condition $E = hf = h\omega/2$

→ Energy Uncertainty: $E - \Delta E < E < E + \Delta E$

Back to Heisenberg's Uncertainty Principle

- $\Delta x . \Delta p \geq h/4\pi \Rightarrow$ If the measurement of the position of a particle is made with a precision Δx and a SIMULTANEOUS measurement of its momentum p_x in the X direction , then the product of the two uncertainties (measurement errors) can never be smaller than $\cong h/4\pi$ irrespective of how precise the measurement tools

- $\Delta E . \Delta t \geq h/4\pi \Rightarrow$ If the measurement of the energy E of a particle is made with a precision ΔE and it took time Δt to make that measurement, then the product of the two uncertainties (measurement errors) can never be smaller than $\cong h/4\pi$ irrespective of how precise the measurement tools

These rules arise from the way we constructed the wave packets describing Matter "pilot" waves

Perhaps these rules are bogus, can we verify this with some physical picture ??

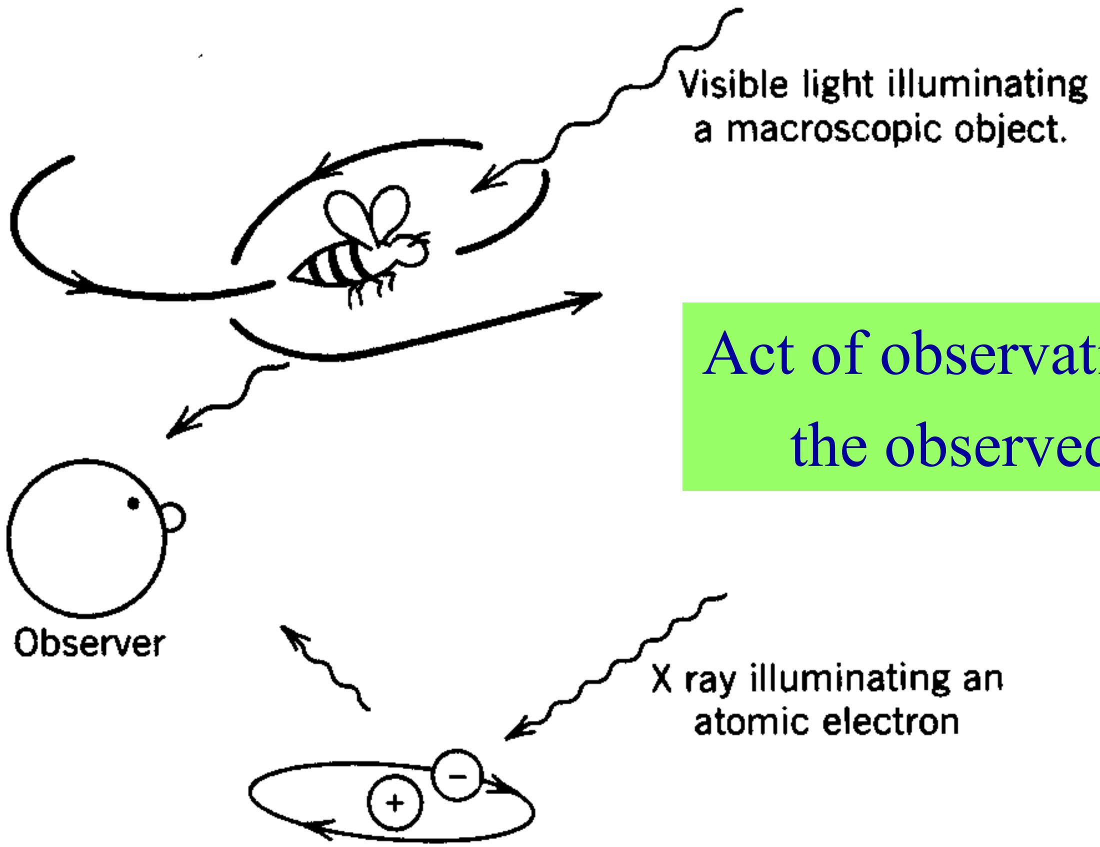
Are You Experienced ?

- What you experience is what you observe
- What you observe is what you measure
- No measurement is perfect, they all have measurement error: question is of the degree
  - Small or large Δ

- Uncertainty Principle and Breaking of Conservation Rules
  - Energy Conservation
  - Momentum Conservation

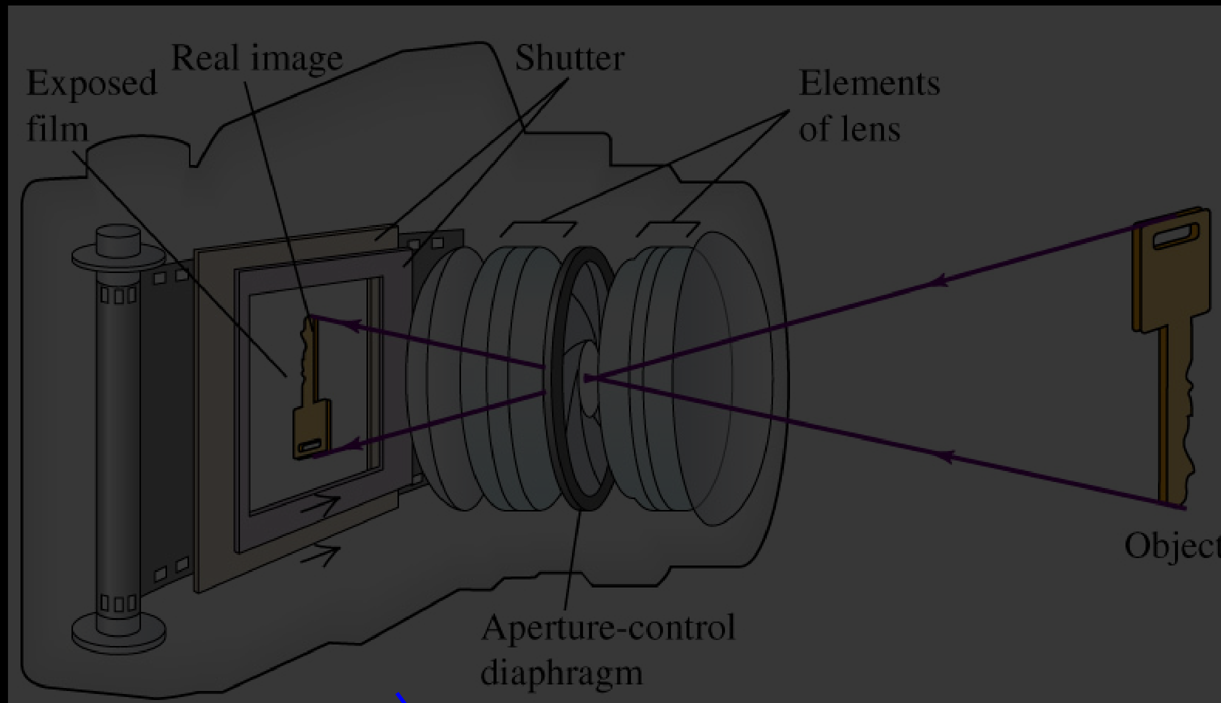
The Act of Observation (Compton Scattering)

Observations of particle motion by means of scattered illumination. When the incident wavelength is reduced to accommodate the size of the particle, the momentum transferred by the photon becomes large enough to disturb the observed motion.

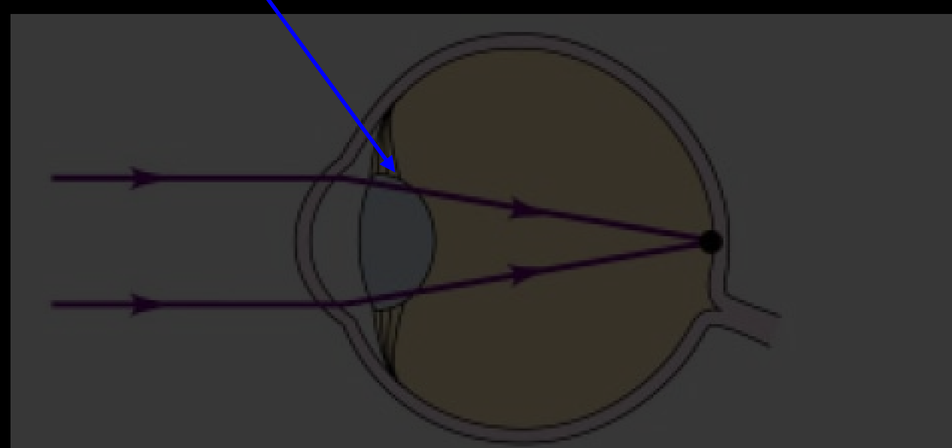


Act of observation disturbs the observed system

Act of Observation Tells All

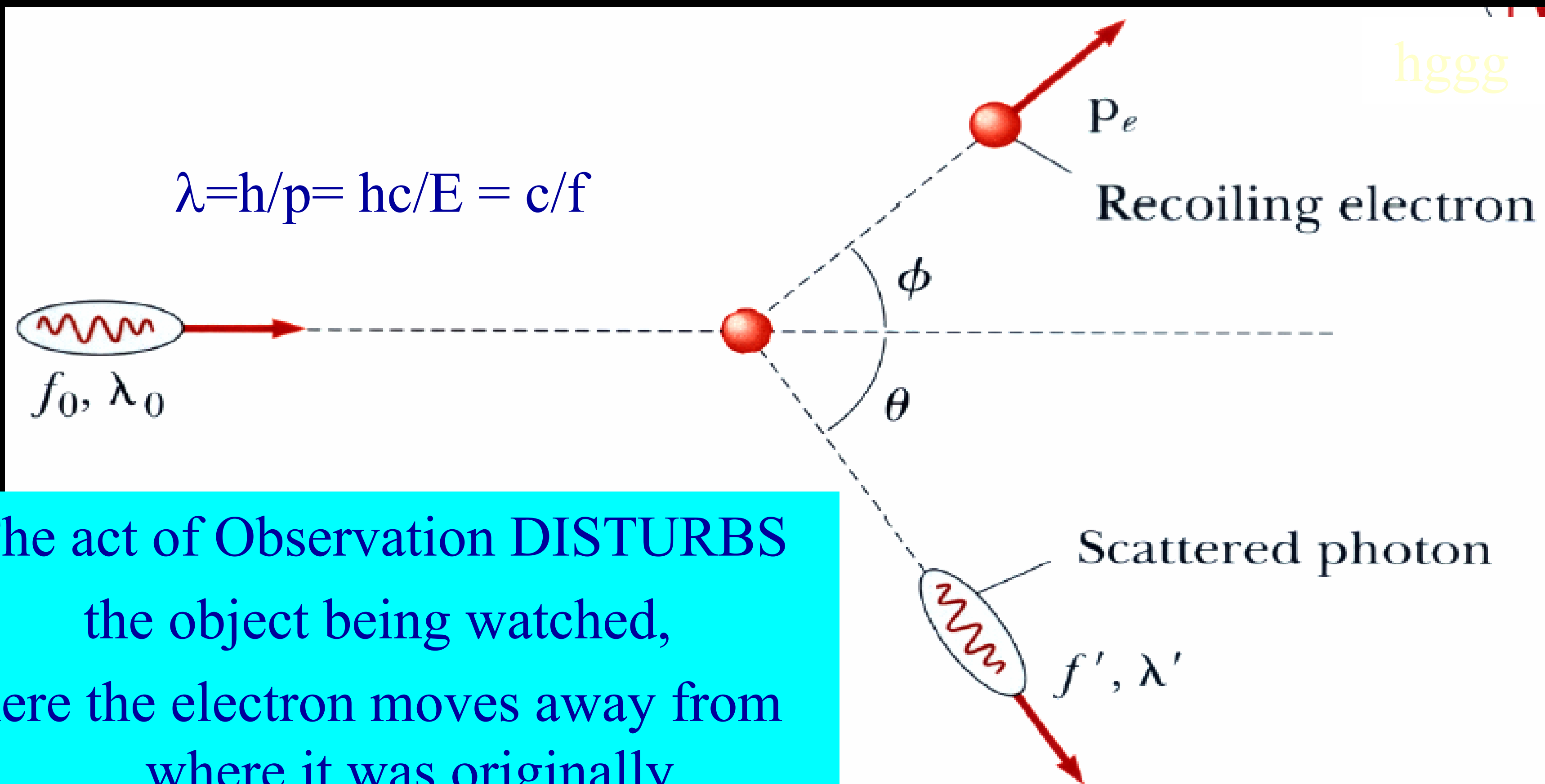


lens



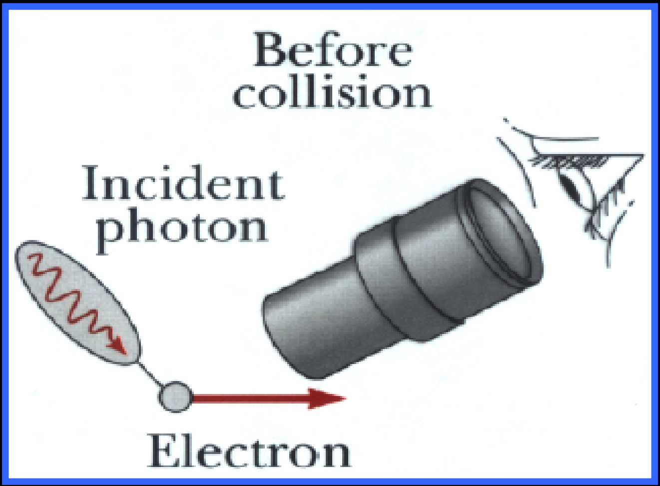
Compton Scattering: Shining light to observe electron

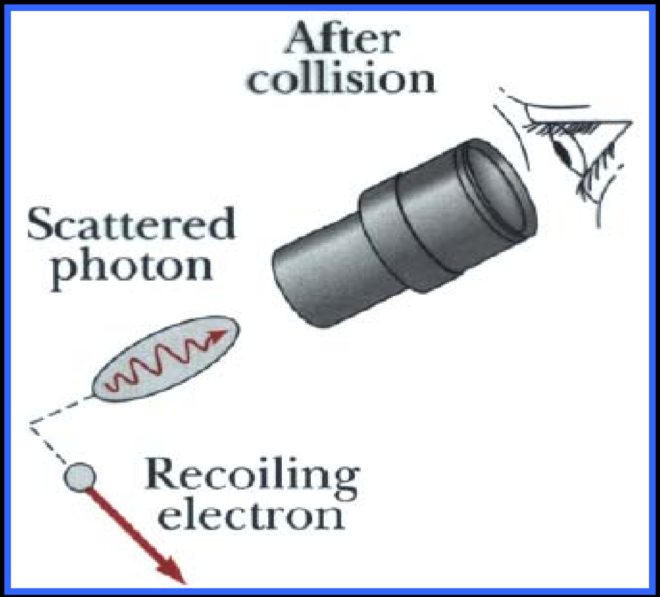
Photon scattering off an electron, Seeing→ the photon enters my eye

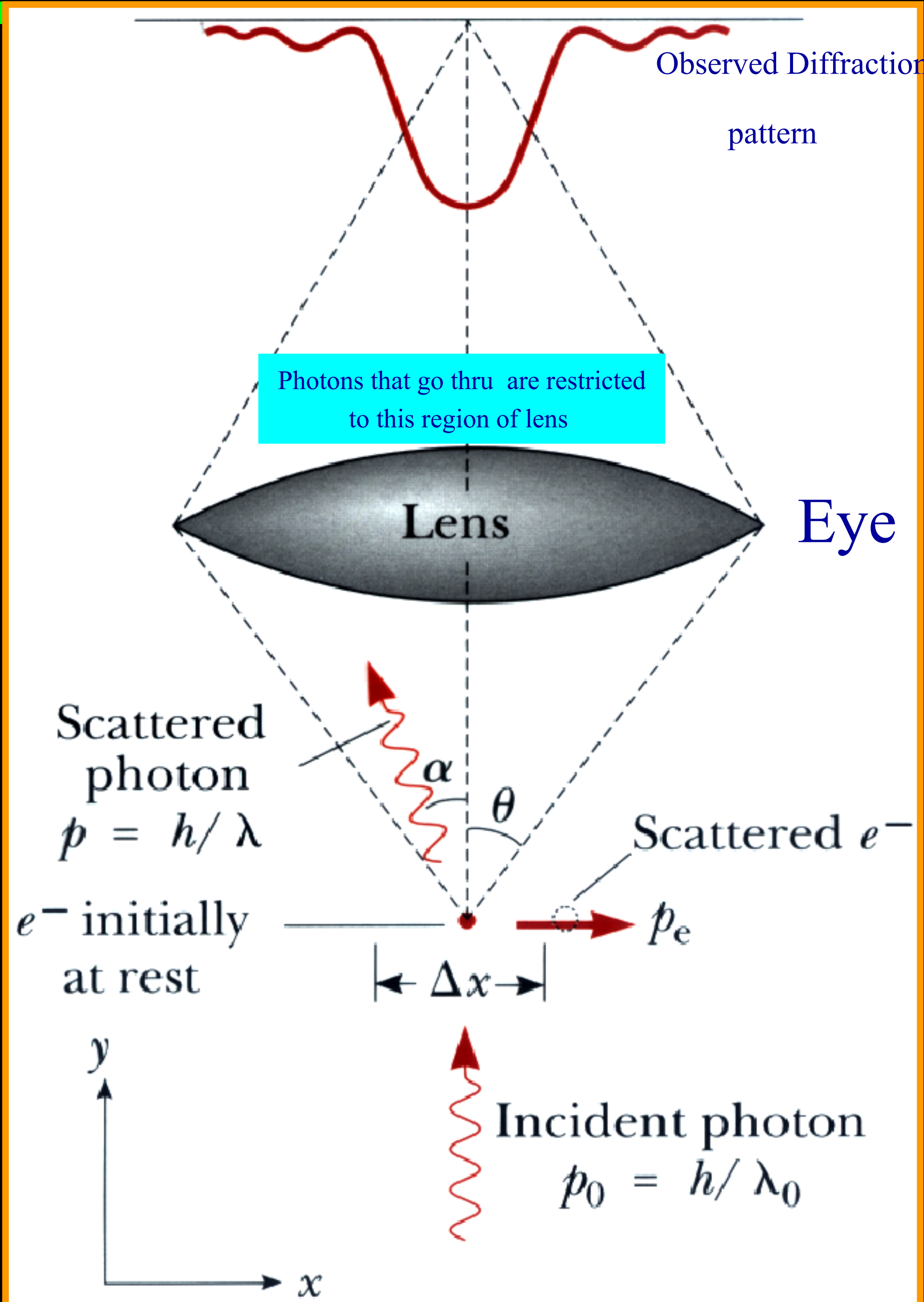


The act of Observation DISTURBS
the object being watched,
here the electron moves away from
where it was originally

Act of Watching: A Thought Experiment







Diffraction By a Circular Aperture (Lens)

See Resnick, Halliday Walker 6th Ed (on S.Reserve), Ch 37, pages 898-900

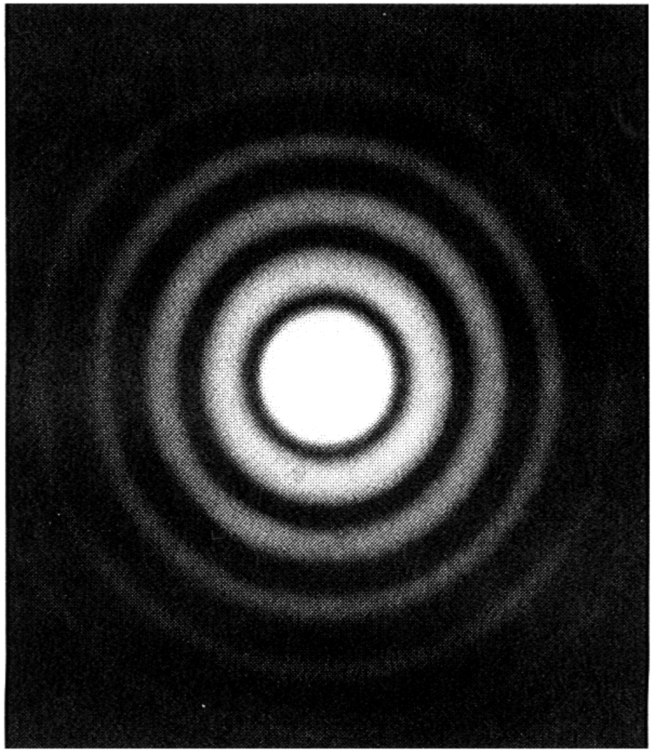


Fig. 37-9 The diffraction pattern of a circular aperture. Note the central maximum and the circular secondary maxima. The figure has been overexposed to bring out these secondary maxima, which are much less intense than the central maximum.

Diffracted image of a point source of light thru a lens (circular aperture of size d)

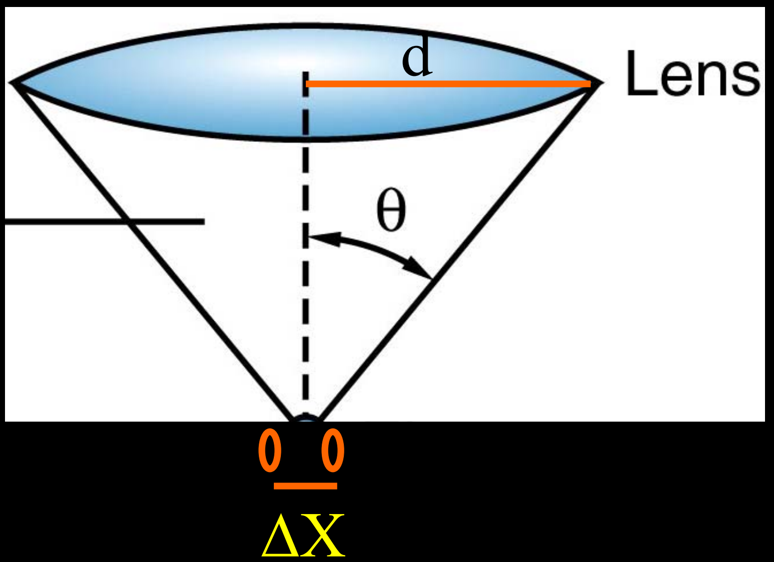
First minimum of diffraction pattern is located by

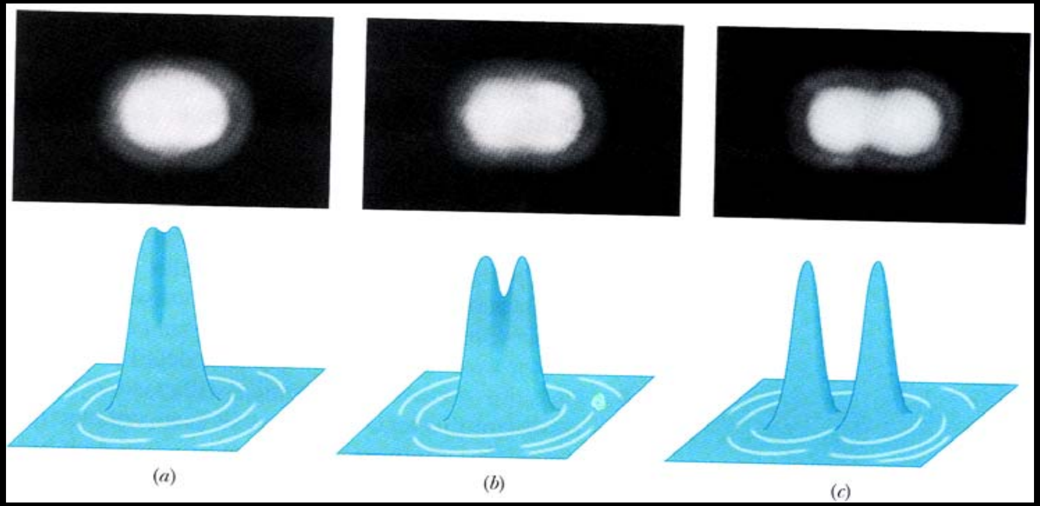
$$\sin\theta = 1.22\frac{\lambda}{d}$$

See previous picture for definitions of ϑ, λ, d

Resolving Power of Light Thru a Lens

Image of 2 separate point sources formed by a converging lens of diameter d, ability to resolve them depends on λ & d because of the Inherent diffraction in image formation



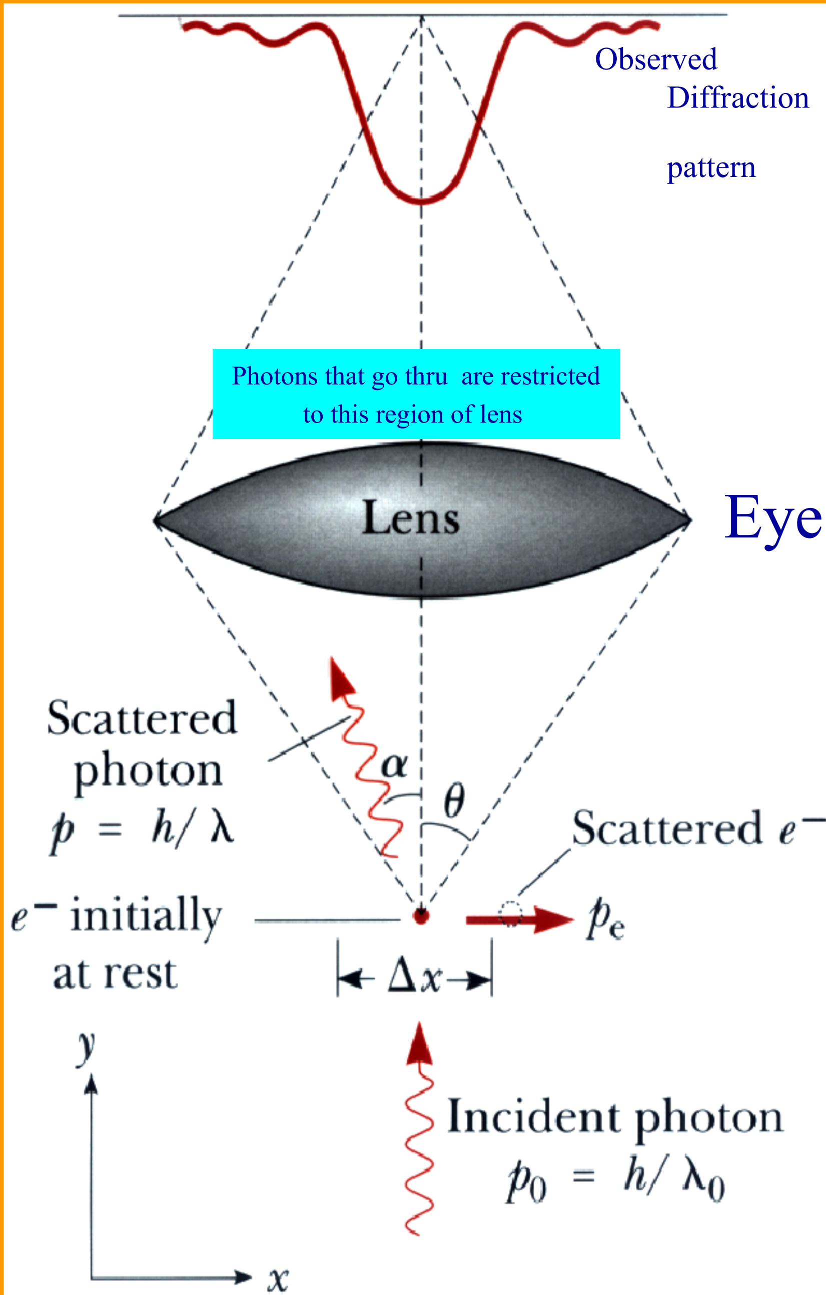


Not resolved   Barely resolved   Resolved

$$\text{Resolving power } \Delta x \simeq \frac{\lambda}{2\sin\theta}$$

θ depends on lens radius d

Putting it all together: Act of Observing an Electron



- Incident light (p,λ) scatters off electron
- To be collected by lens → γ must scatter thru angle α
  - $-\vartheta \leq \alpha \leq \vartheta$
- Due to Compton scatter, electron picks up momentum
  - P_X , P_Y

$$-\frac{h}{\lambda}\sin\theta \leq P_x \leq \frac{h}{\lambda}\sin\theta$$

electron momentum uncertainty is

$$\Delta p \cong \frac{\sim 2h}{\lambda}\sin\theta$$

- After passing thru lens, photon diffracts, lands somewhere on screen, image (of electron) is fuzzy
- How fuzzy ? Optics says shortest distance between two resolvable points is :

$$\Delta x = \frac{\lambda}{2\sin\theta}$$

- Larger the lens radius, larger the $\vartheta \Rightarrow$ better resolution

$$\Rightarrow \Delta p.\Delta x \simeq \left(\frac{2h\sin\theta}{\lambda}\right)\left(\frac{\lambda}{2\sin\theta}\right) = h$$

$$\Rightarrow \Delta p.\Delta x \geq \hbar/2$$

Aftermath of Uncertainty Principle

- Deterministic (Newtonian) physics topples over
  - Newton's laws told you all you needed to know about trajectory of a particle
    - Apply a force, watch the particle go !
      - Know every thing ! X, v, p , F, a
      - Can predict exact trajectory of particle if you had perfect device

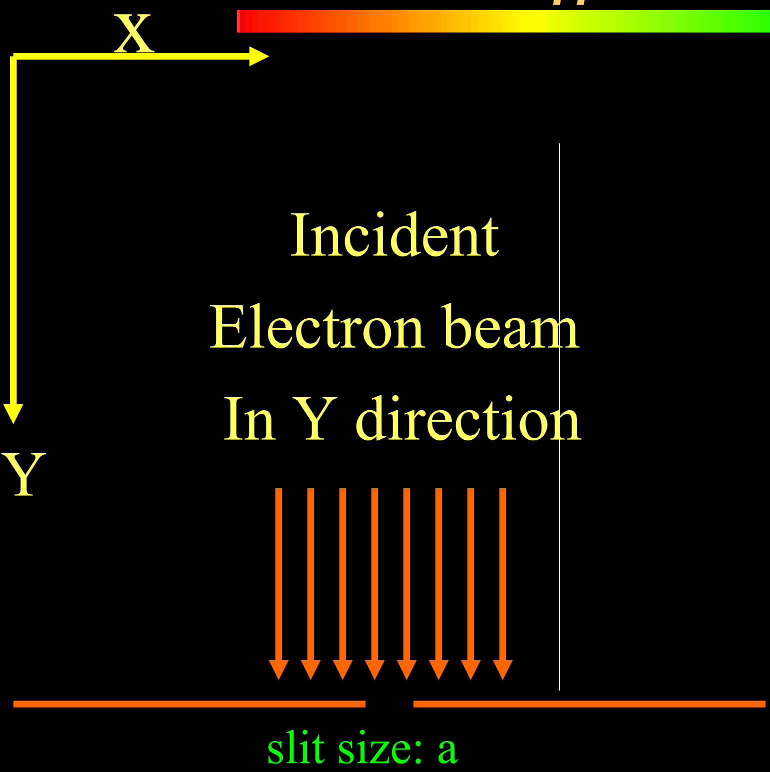
- No so in the subatomic world !
  - Of small momenta, forces, energies
  - Can't predict anything exactly
    - Can only predict probabilities
      - There is so much chance that the particle landed here or there
      - Cant be sure !....cognizant of the errors of thy observations

All Measurements Have Associated Errors

- If your measuring apparatus has an intrinsic inaccuracy (error) of amount Δp
- Then results of measurement of momentum p of an object at rest can easily yield a range of values accommodated by the measurement imprecision :
  - $-\Delta p \leq p \leq \Delta p$: you will measure any of these values for the momentum of the particle

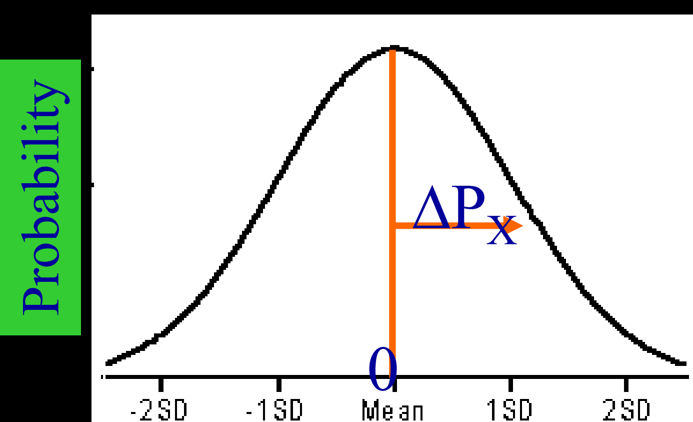
- Similarly for all measurable quantities like x, t, Energy !

Matter Diffraction & Uncertainty Principle



Momentum measurement beyond slit show particle not moving exactly in Y direction, develops a X component Of motion $-\Delta p_x \le p_x \le \Delta p_x$ **with**

$$\Delta p_X = h/(2\pi\, a)$$

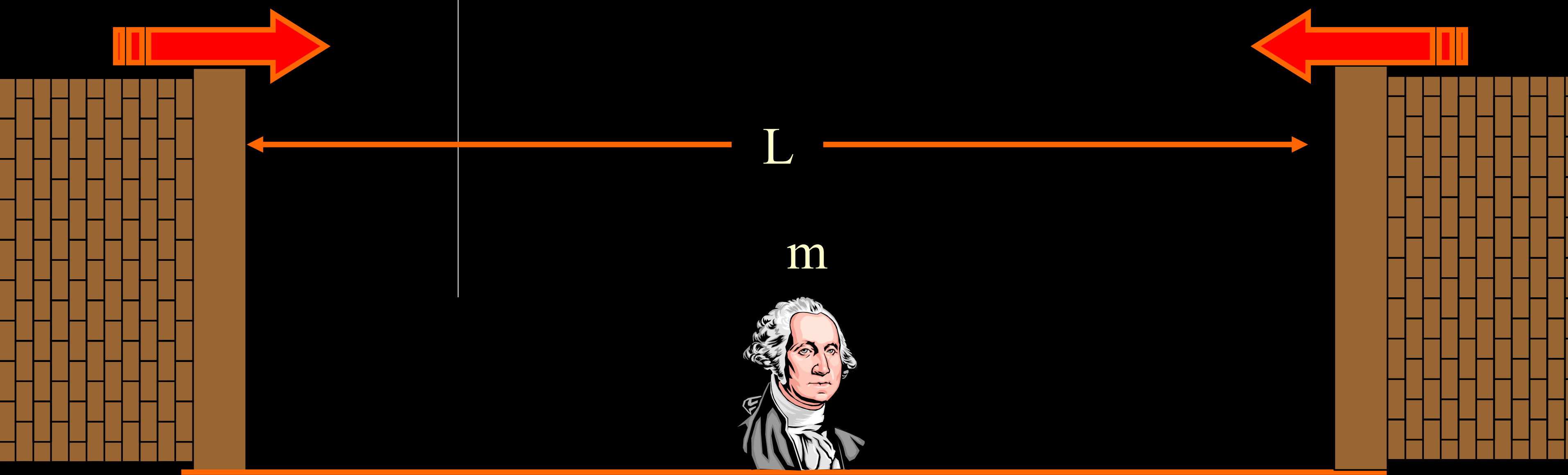


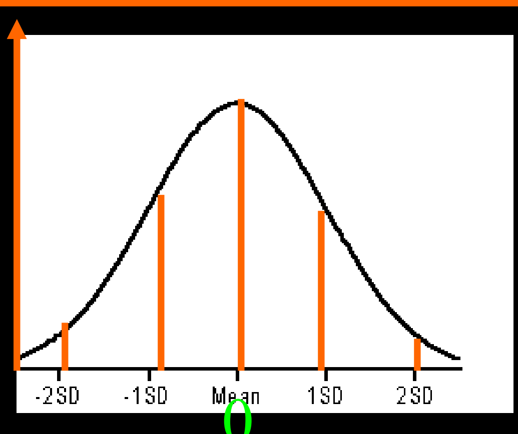
X component P_X of momentum

Particle at Rest Between Two Walls

Object of mass M at rest between two walls originally at infinity

What happens to our perception of George's momentum as the walls are brought in ?





George's Momentum p

On average, measure <p> = 0

but there are quite large fluctuations!

Width of Distribution $= \Delta P$

$$\Delta P = \sqrt{(P^2)_{ave} - (P_{ave})^2}; \quad \Delta P \sim \frac{\hbar}{L}$$