



# *4E : The Quantum Universe*

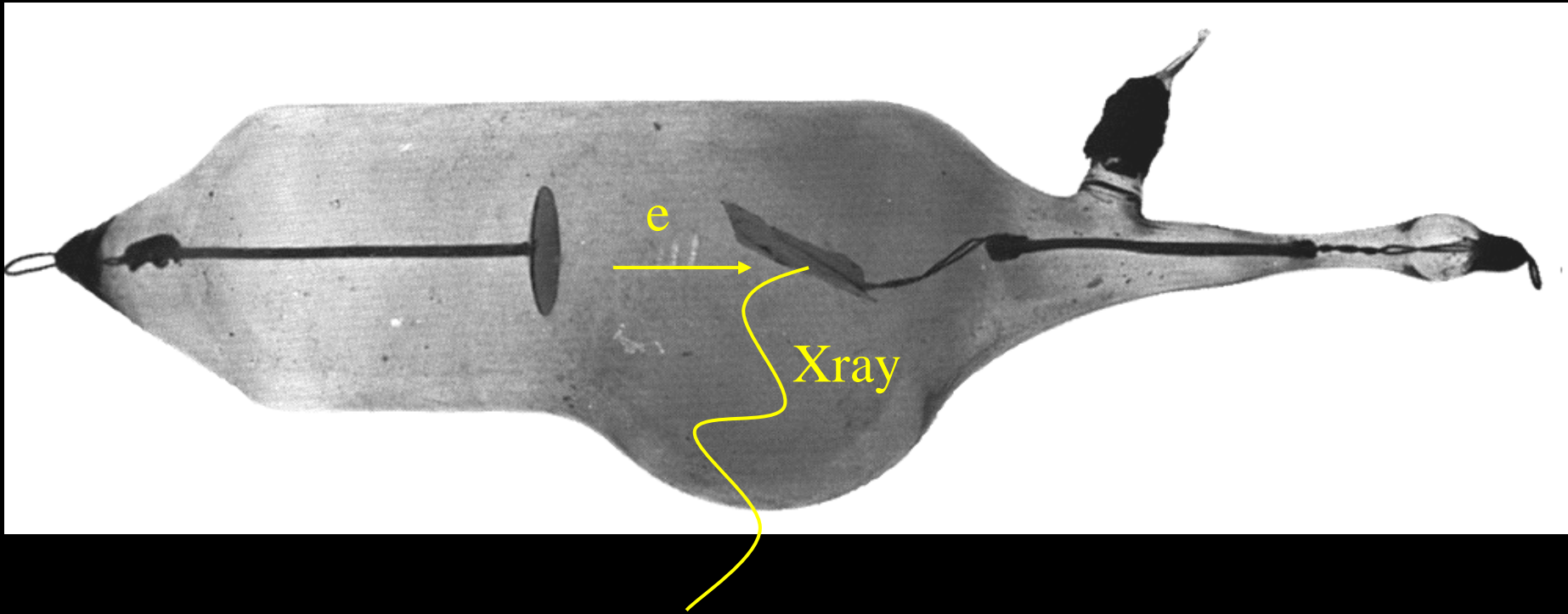


Lecture 5, April 5

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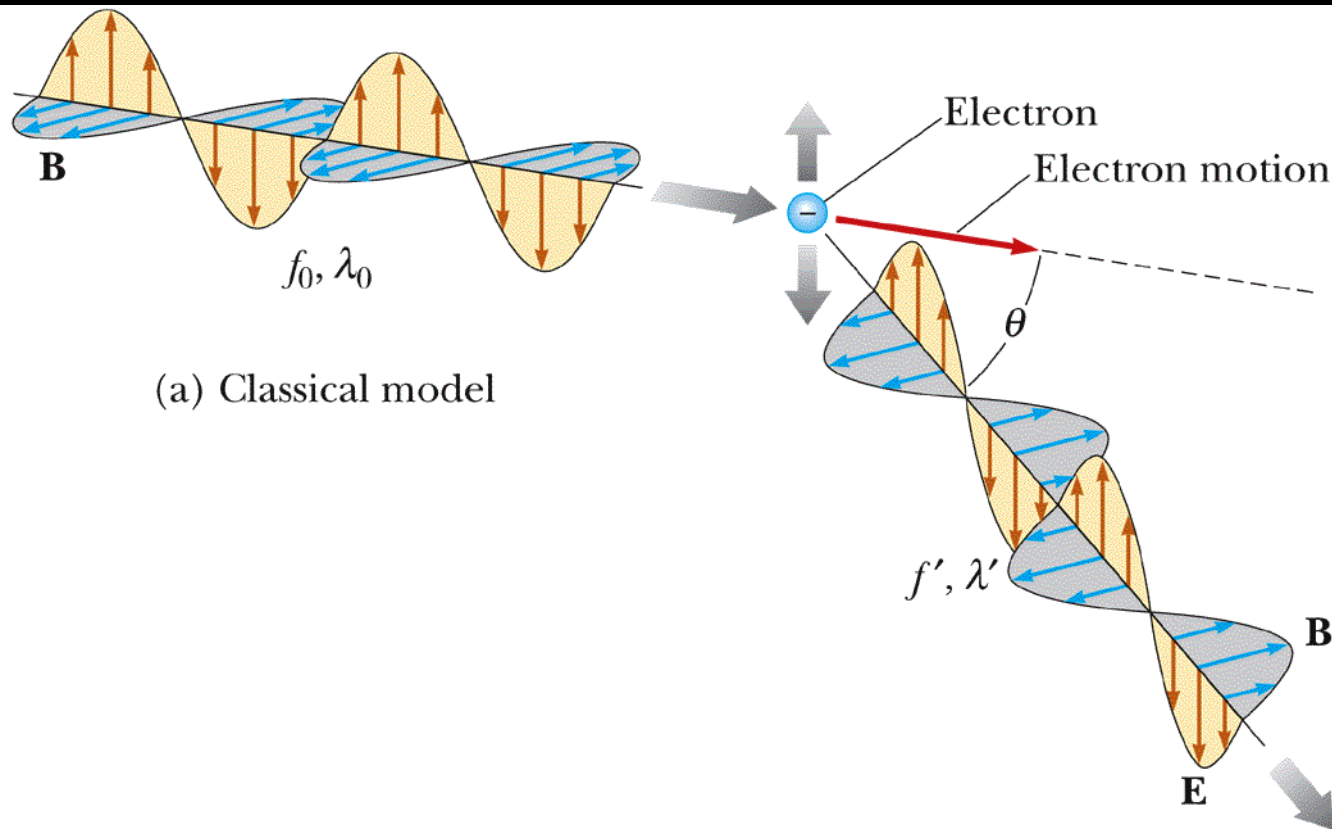
# *An X-ray Tube from 20<sup>th</sup> Century*



The “High Energy Accelerator” of 1900s:  
produced energetic light : X –Ray , gave new optic  
to subatomic world

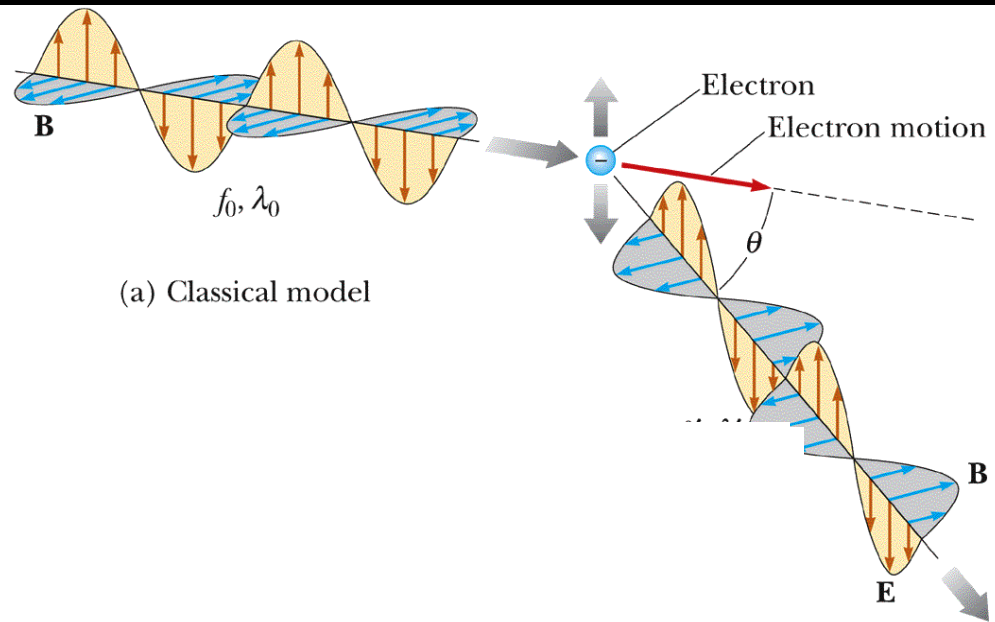
# Compton Scattering : Quantum Pool !

- Arthur Compton (USA) proves that X-rays (EM Waves) have particle like properties (acts like photons)
  - Showed that classical theory failed to explain the scattering effect of X rays on to free (not bound, barely bound electrons)
- Experiment : shine X ray on to a surface with “almost” free electrons
  - Watch the scattering of light off electron : measure time +  $\lambda$  of scattered X-ray

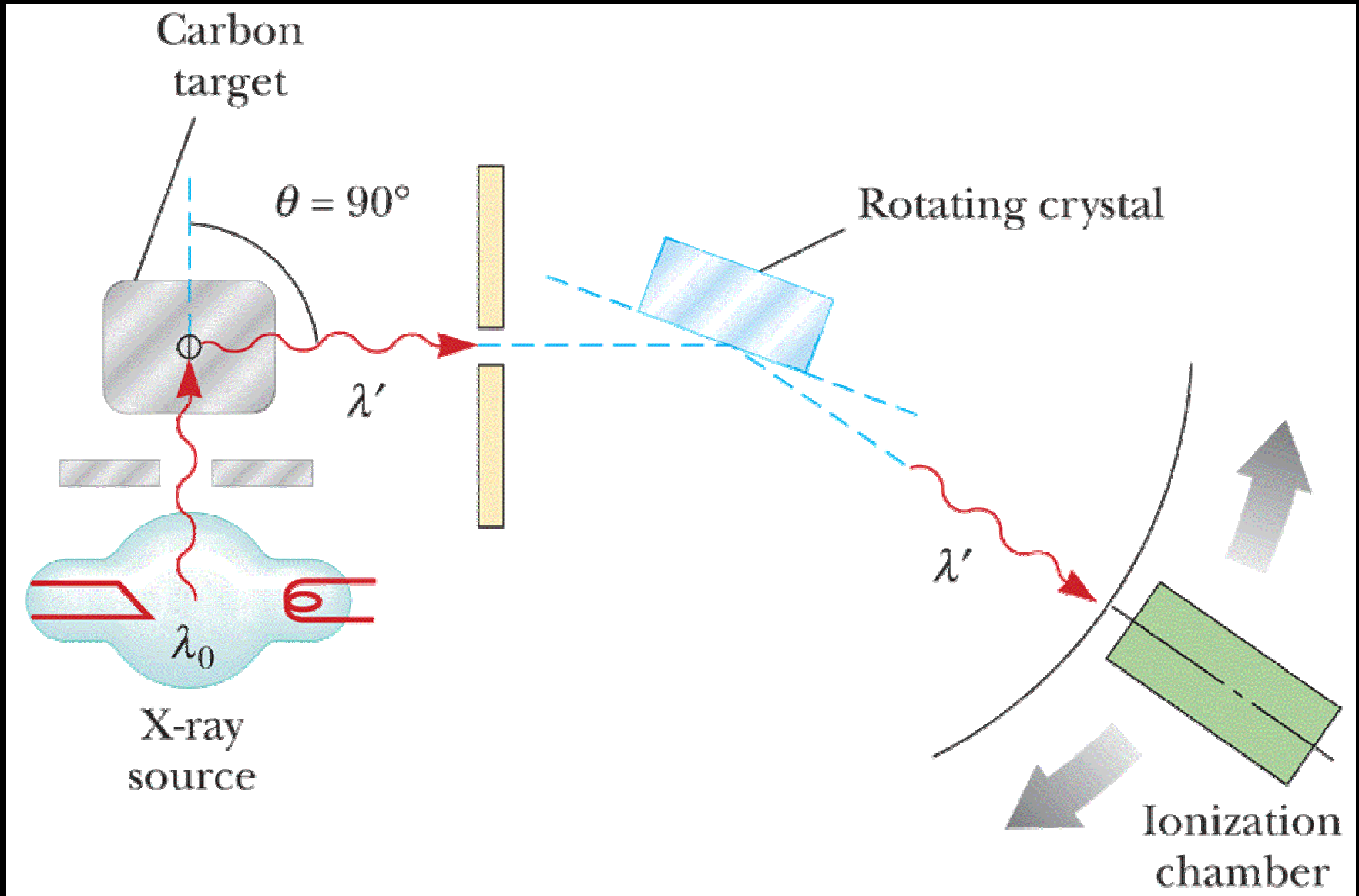


# Compton Effect: what should Happen Classically?

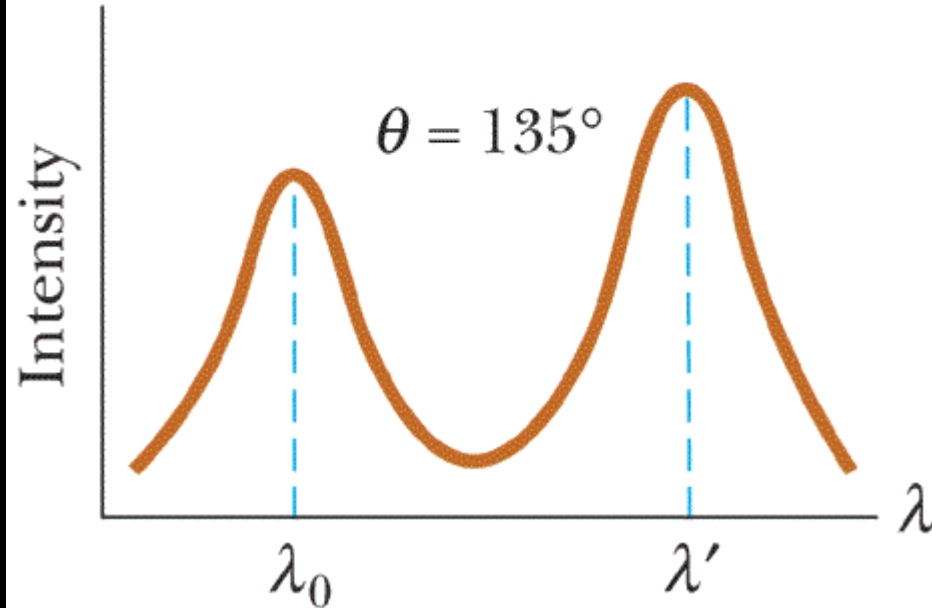
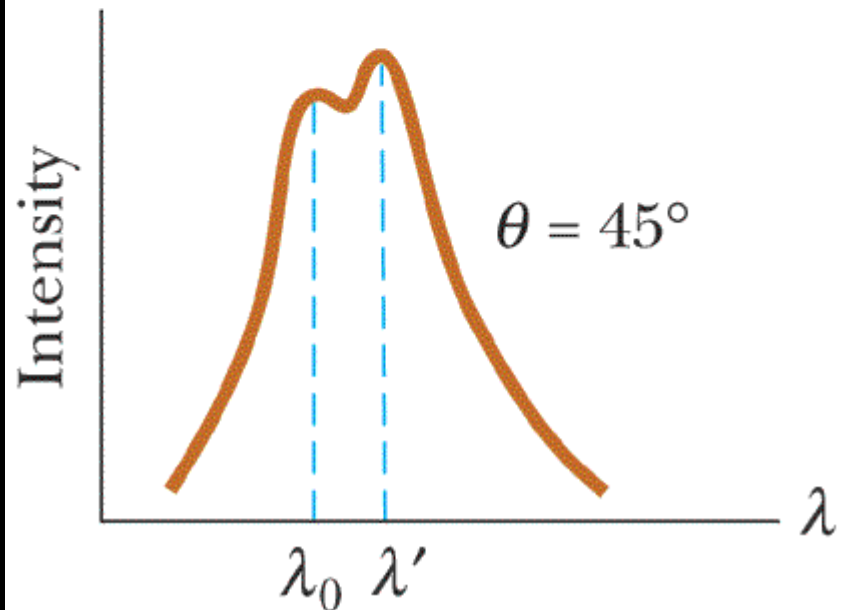
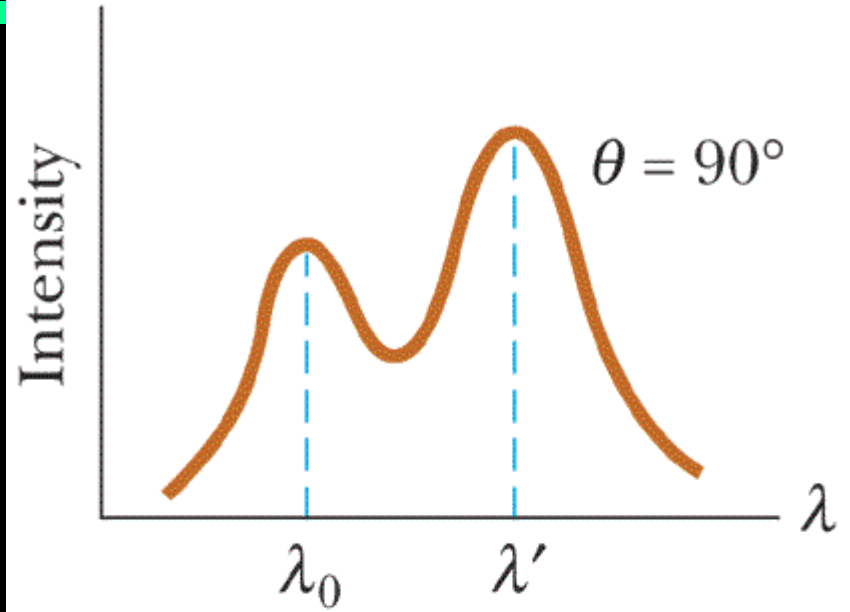
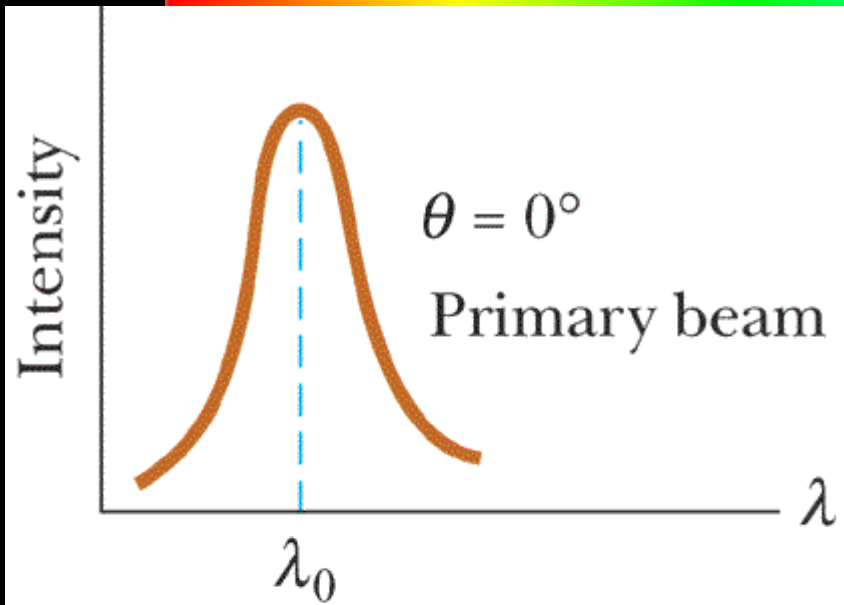
- Plane wave  $[f, \lambda]$  incident on a surface with loosely bound electrons  $\rightarrow$  interaction of E field of EM wave with electron:  $\mathbf{F} = e\mathbf{E}$
- Electron oscillates with  $f = f_{\text{incident}}$
- Eventually radiates spherical waves with  $f_{\text{radiated}} = f_{\text{incident}}$ 
  - At all scattering angles,  $\Delta f$  &  $\Delta \lambda$  must be zero
- Time delay while the electron gets a “tan”: soaks in radiation



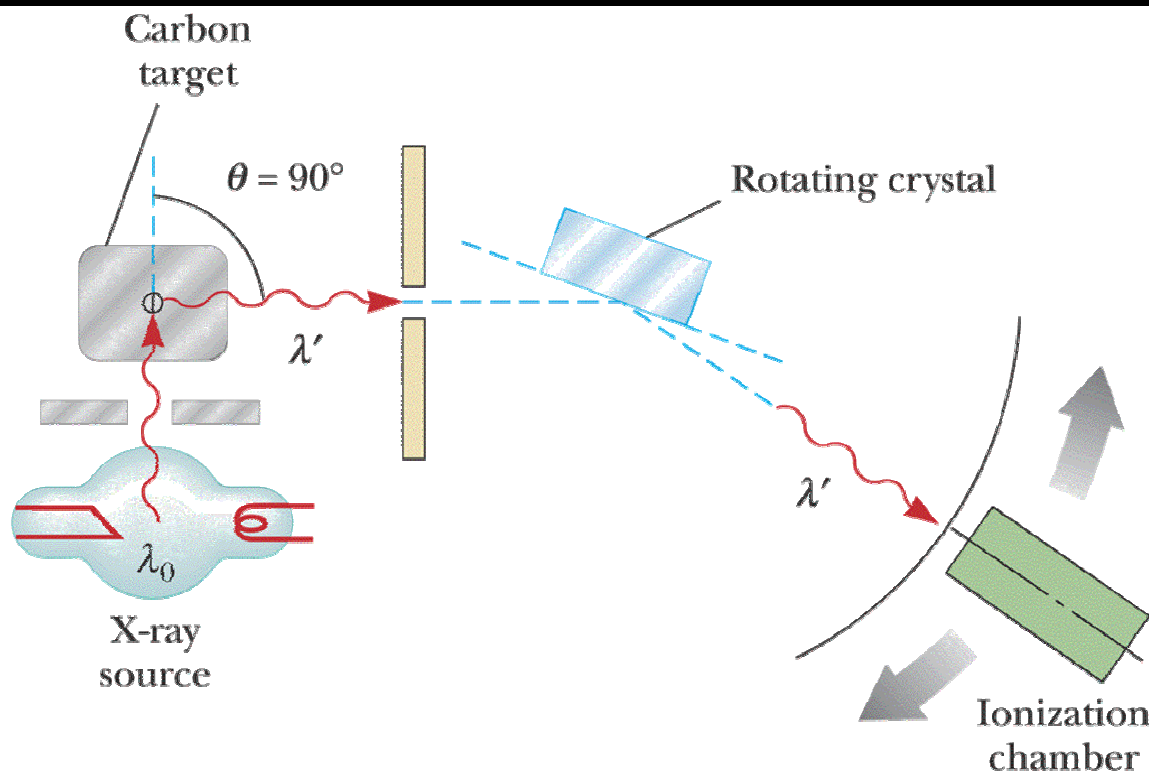
# Compton Scattering : Experimental Setup & Results



# Compton Scattering : Observations



# Compton Scattering : Summary of Observations

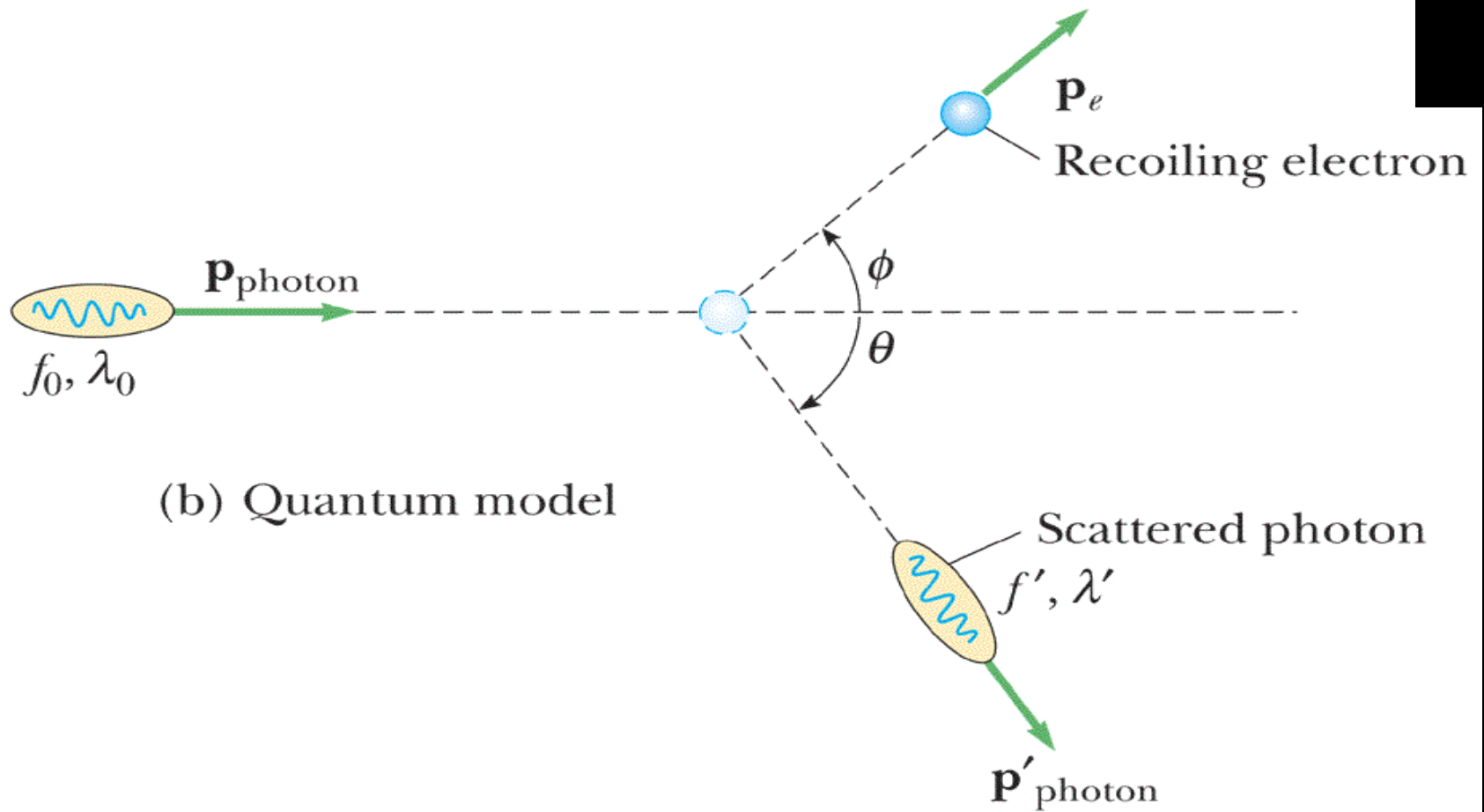


$$\Delta\lambda = (\lambda' - \lambda) \propto (1 - \cos \theta) !$$

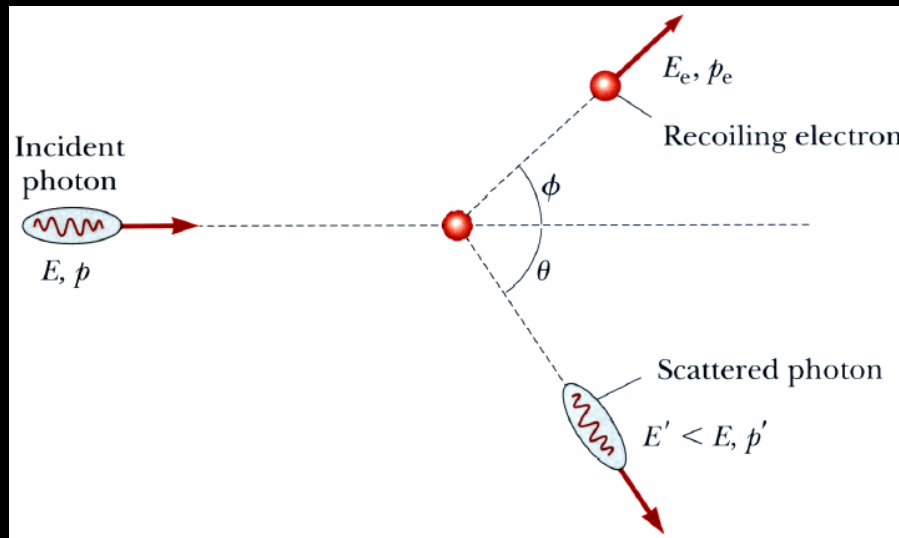
Not isotropy in distribution of scattered radiation

How does one explain this startling anisotropy?

# Compton Effect : Quantum (Relativistic) Pool



# Compton Scattering: The Quantum Picture



**Energy Conservation:**

$$E + m_e c^2 = E' + E_e$$

**Momentum Conserv:**

$$p = p' \cos \theta + p_e \cos \phi$$

$$0 = p' \sin \theta - p_e \sin \phi$$

Use these to **eliminate electron deflection angle** (not measured)



$$p_e \cos \phi = p - p' \cos \theta$$

$$p_e \sin \phi = p' \sin \theta$$

Square and add  $\Rightarrow$

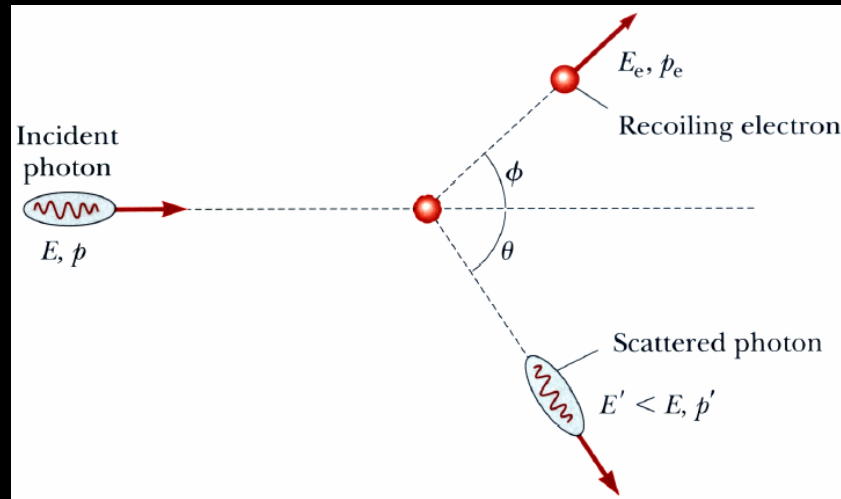
$$p_e^2 = p^2 - 2pp' \cos \theta + p'^2$$

Eliminate  $p_e$  &  $E_e$  using

$$E_e^2 = p_e^2 c^2 + m_e^2 c^4 \quad \&$$

$$E_e = (E - E') + m_e c^2$$

# Compton Scattering: The Quantum Picture



$$\left( (E - E') + m_e c^2 \right)^2 = \left[ p^2 - 2pp' \cos \theta + p'^2 \right] + (m_e c^2)^2$$

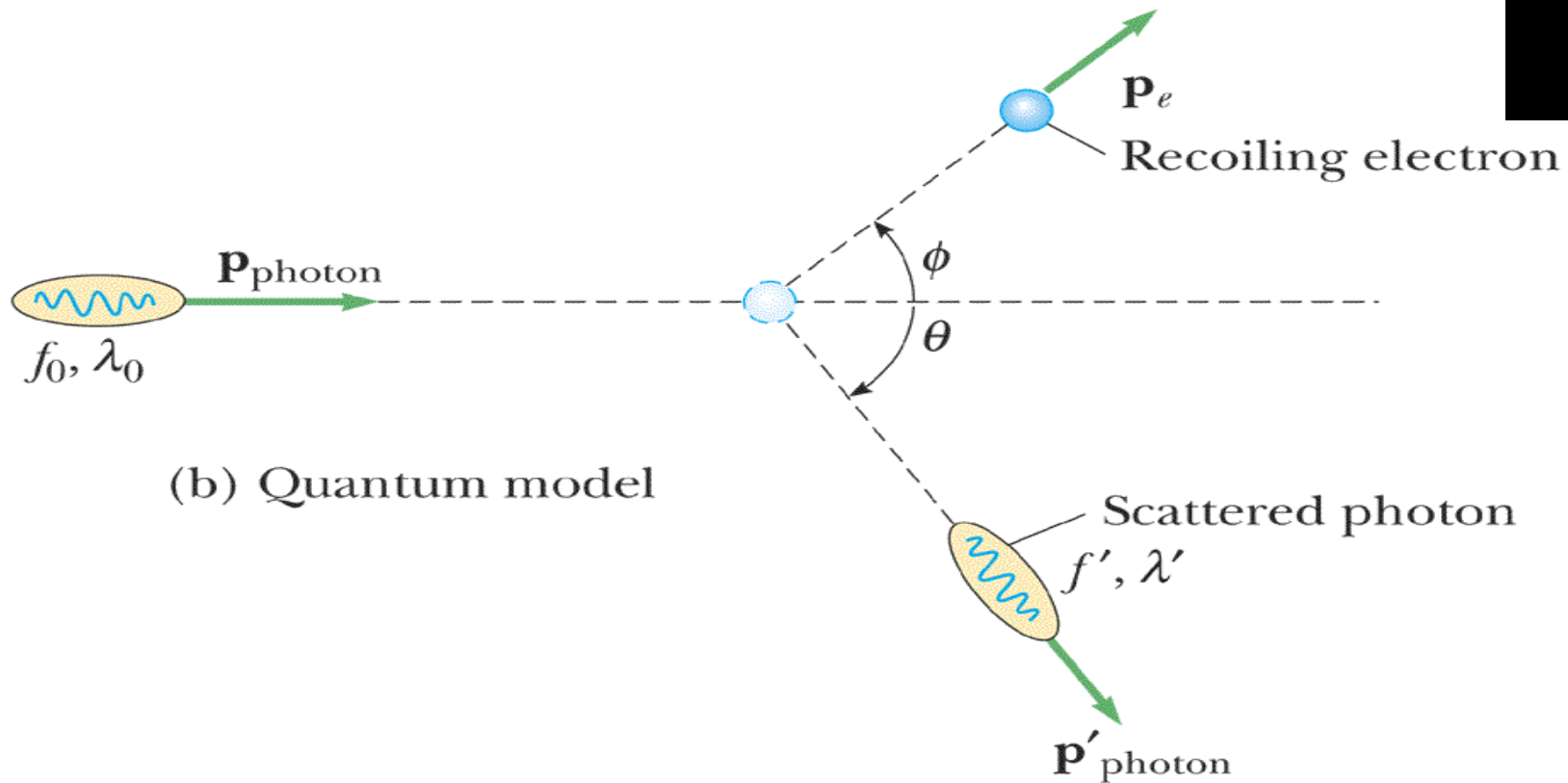
For light  $p = \frac{E}{c} \Rightarrow$

$$E^2 + E'^2 - 2EE' + 2(E - E')mc^2 = \left[ \frac{E^2}{c^2} - 2\frac{EE'}{c^2} \cos \theta + \frac{E'^2}{c^2} \right] c^2$$

$$\Rightarrow -EE' + (E - E')mc^2 = -EE' \cos \theta$$

$$\Rightarrow \frac{E - E'}{EE'} = -\frac{1}{m_e c^2} (1 - \cos \theta) \Rightarrow \boxed{(\lambda' - \lambda) = \left( \frac{h}{m_e c} \right) (1 - \cos \theta)}$$

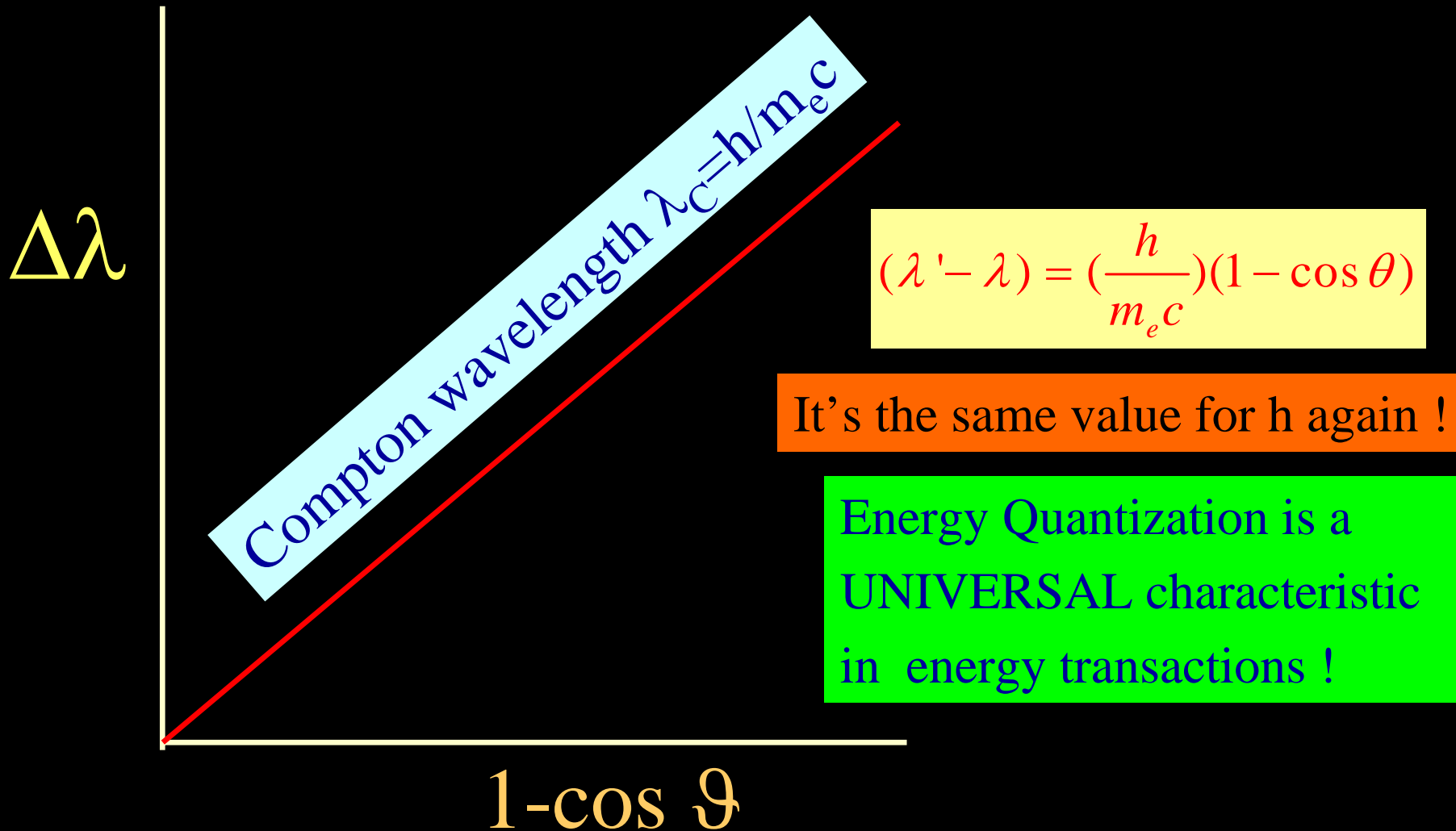
# Rules of Quantum Pool between Photon and Electron



$$(\lambda' - \lambda) = \left( \frac{h}{m_e c} \right) (1 - \cos \theta)$$

# Checking for “h” in Compton Scattering

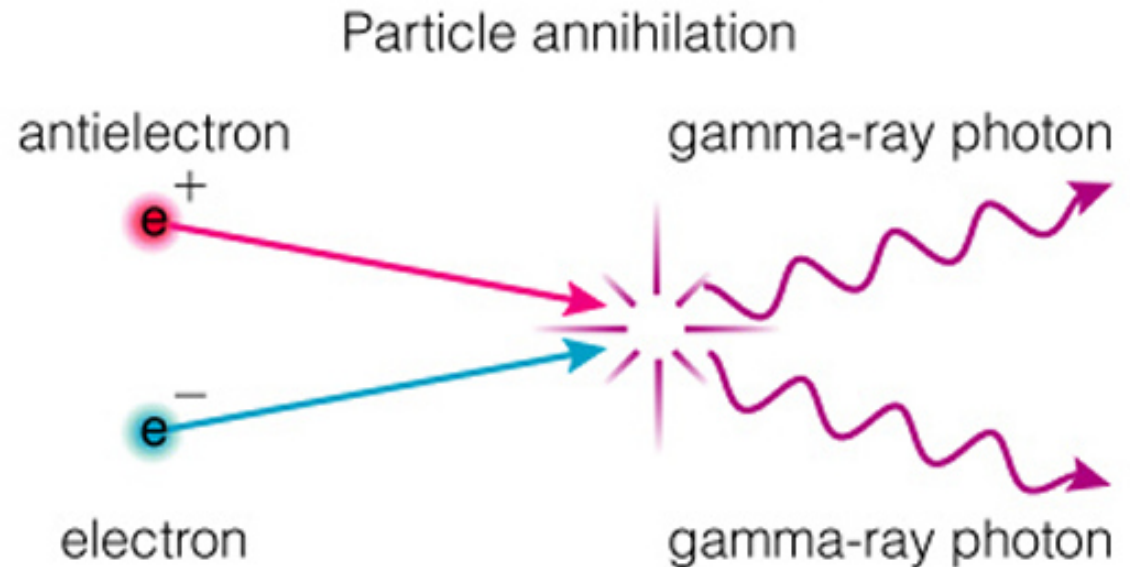
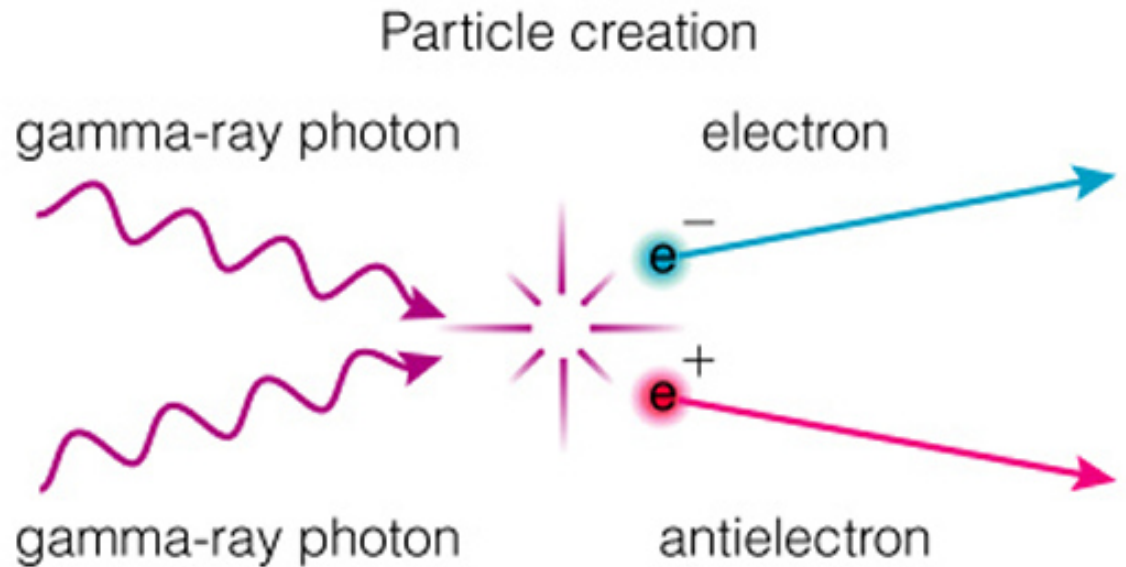
From scattered photon  $\lambda$ , plot  $\Delta\lambda$ , calculate slope and measure “h”



# Other forms of Interaction of Energy Exchange between Radiation and Matter

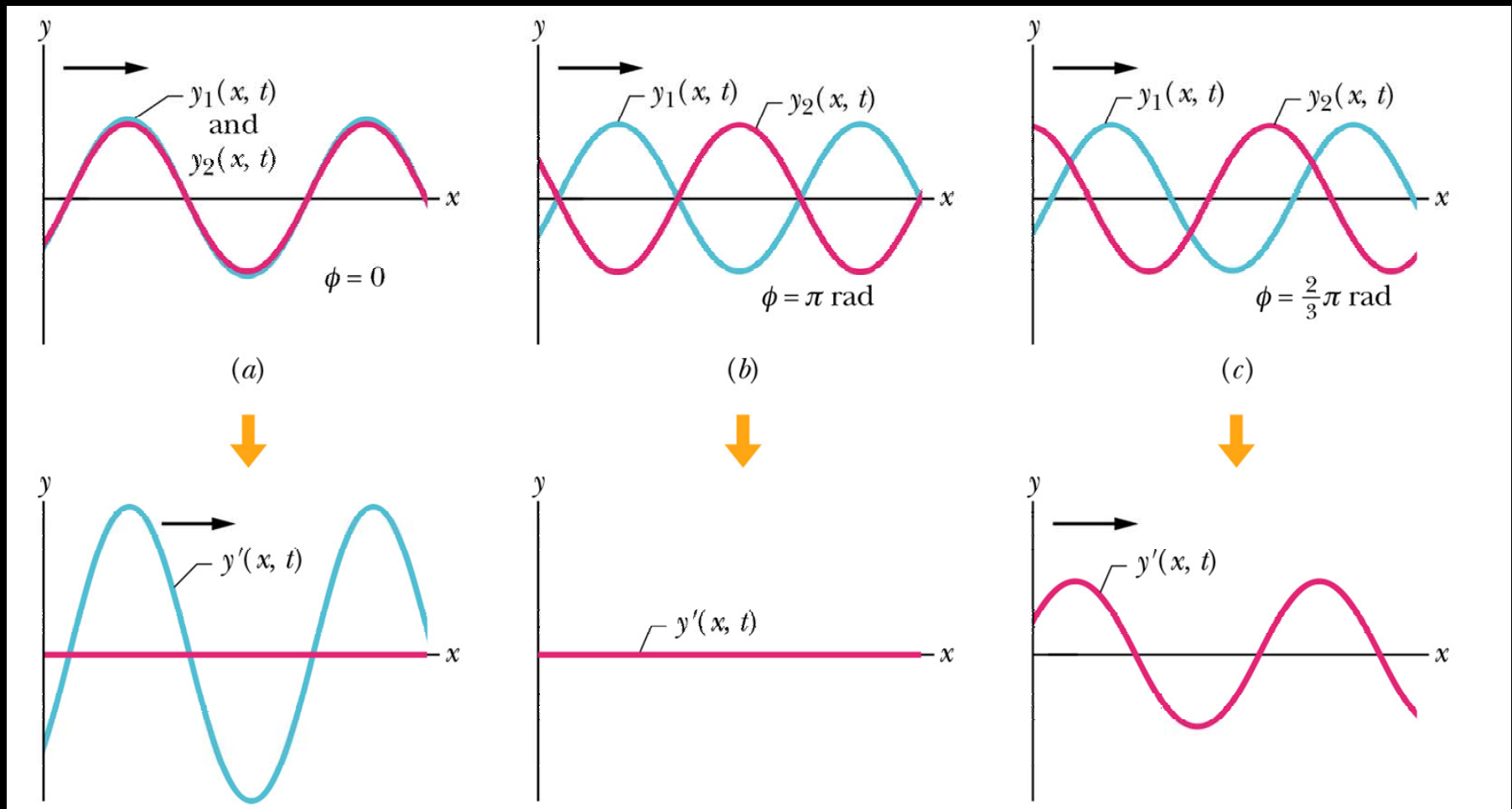
$$E \rightleftharpoons mc^2 + mc^2$$

Always same form of  
Matter & Antimatter

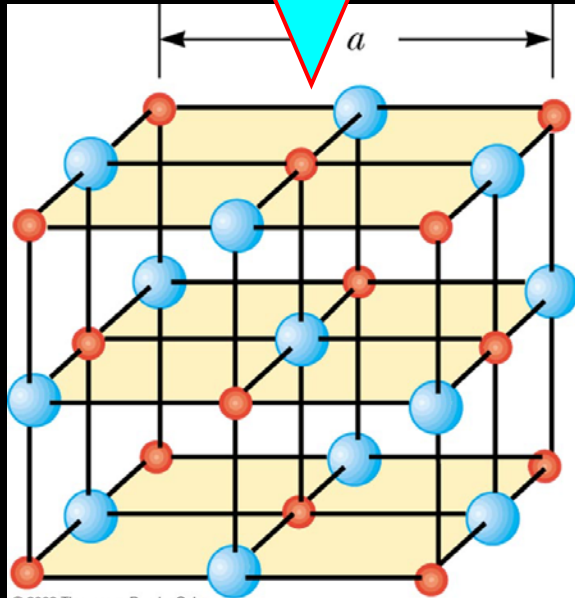
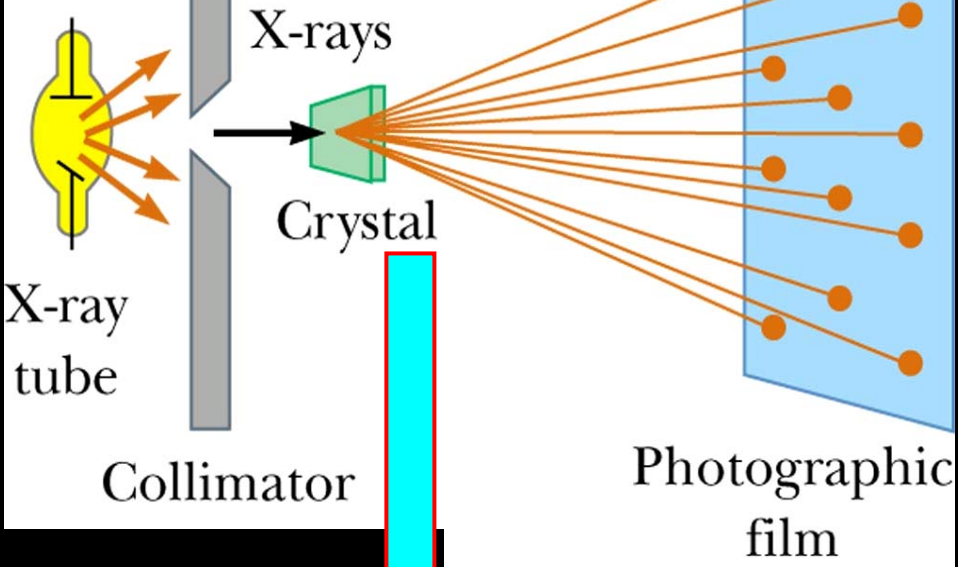


# Constructive Interference depends on Path (or phase) diff. Traversed

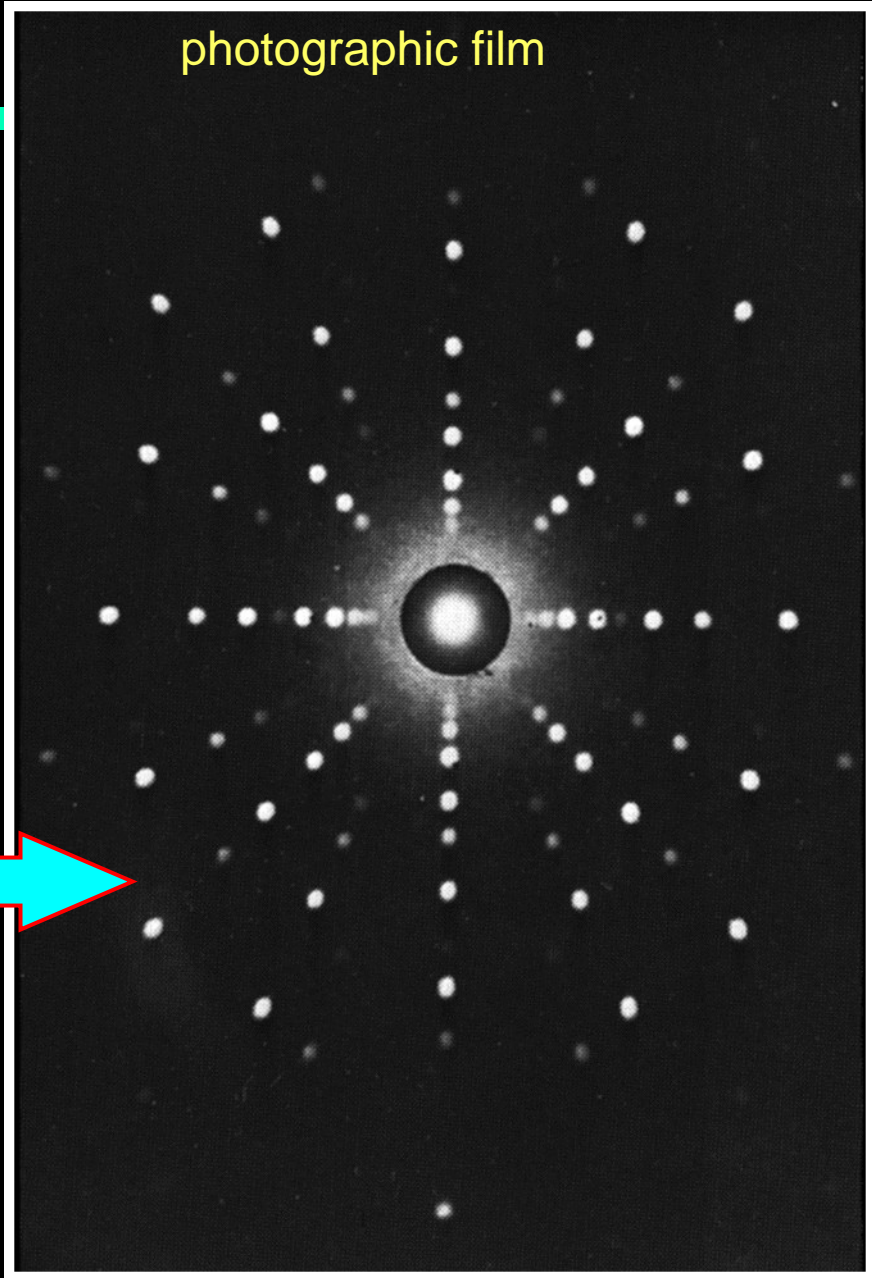
Two Identical waves  $y_i(x,t) = y_{\max} \sin(k_i x - \omega_i t + \phi_i)$  travel along +x and interfere to give a resulting wave  $y'(x,t)$ . The resulting wave form depends on relative phase difference between 2 waves. Shown for  $\Delta\phi = 0, \pi, \frac{2}{3}\pi$



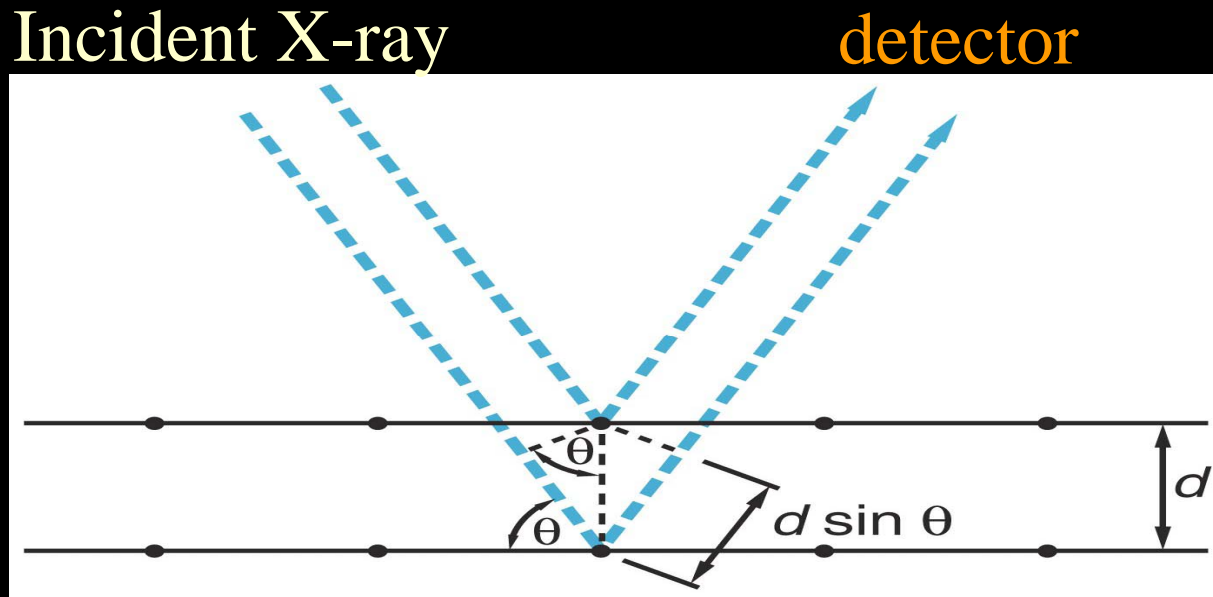
# Bragg Scattering



photographic film



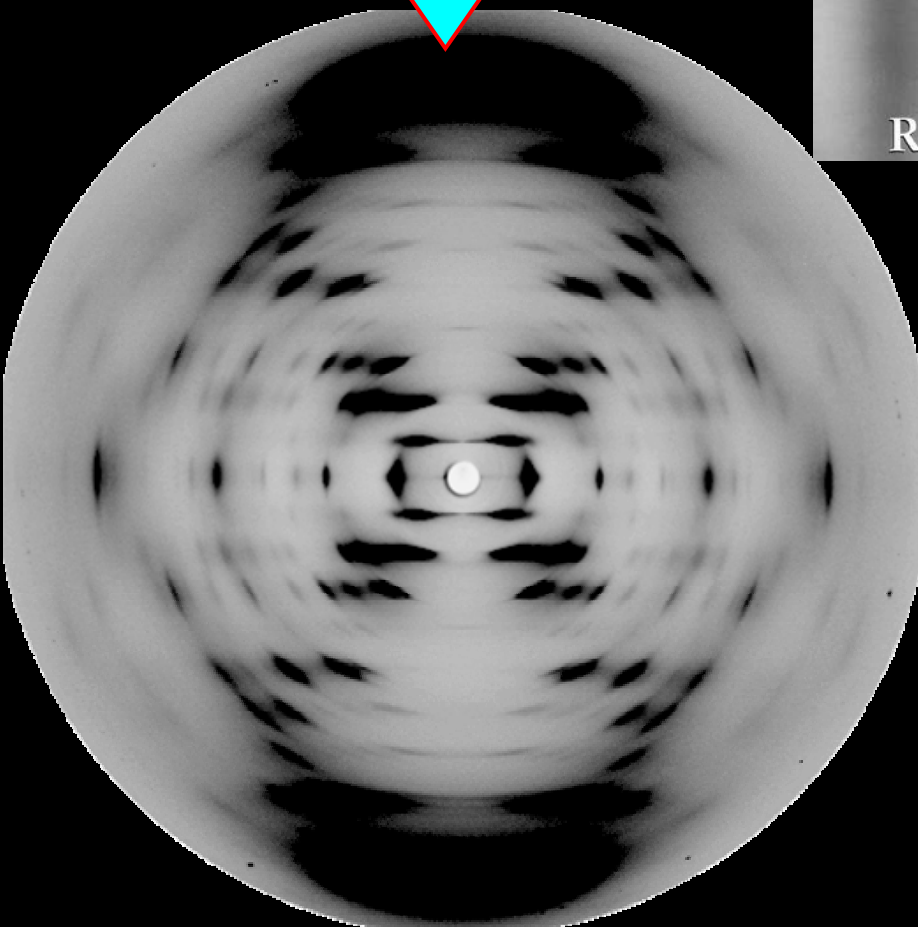
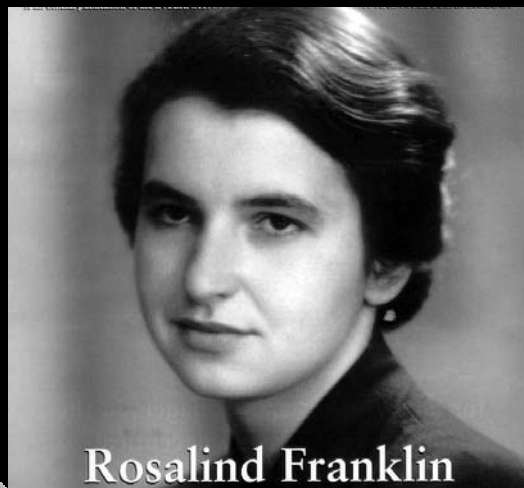
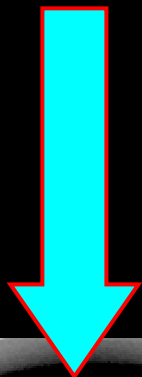
## Bragg Scattering: Probing Atoms With X-Rays



Constructive Interference when net phase difference is  $0, 2\pi$  etc  
This implied path difference traveled by two waves must be integral  
multiple of wavelength :  $n\lambda=2d\sin\theta$

*From X Ray (EM Wave) Scattering data,  
size of atoms was known to be about  $10^{-10} m$*

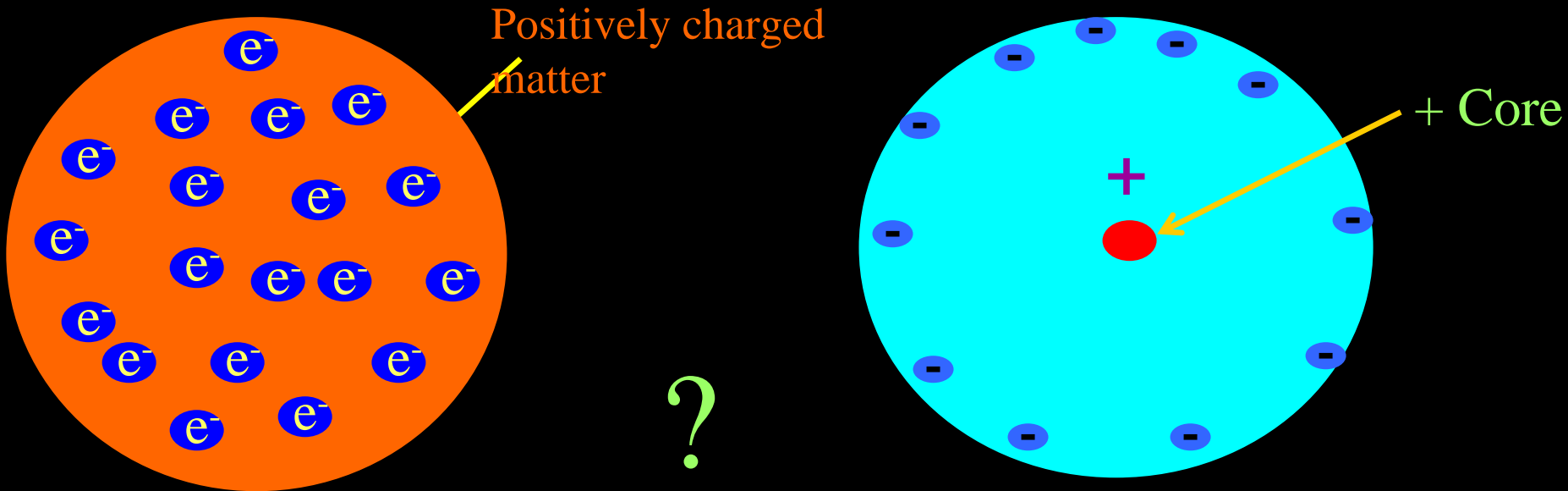
*X-Ray Picture of a DNA Crystal and Discovery of DNA Structure !*



# Where are the electrons inside the atom?



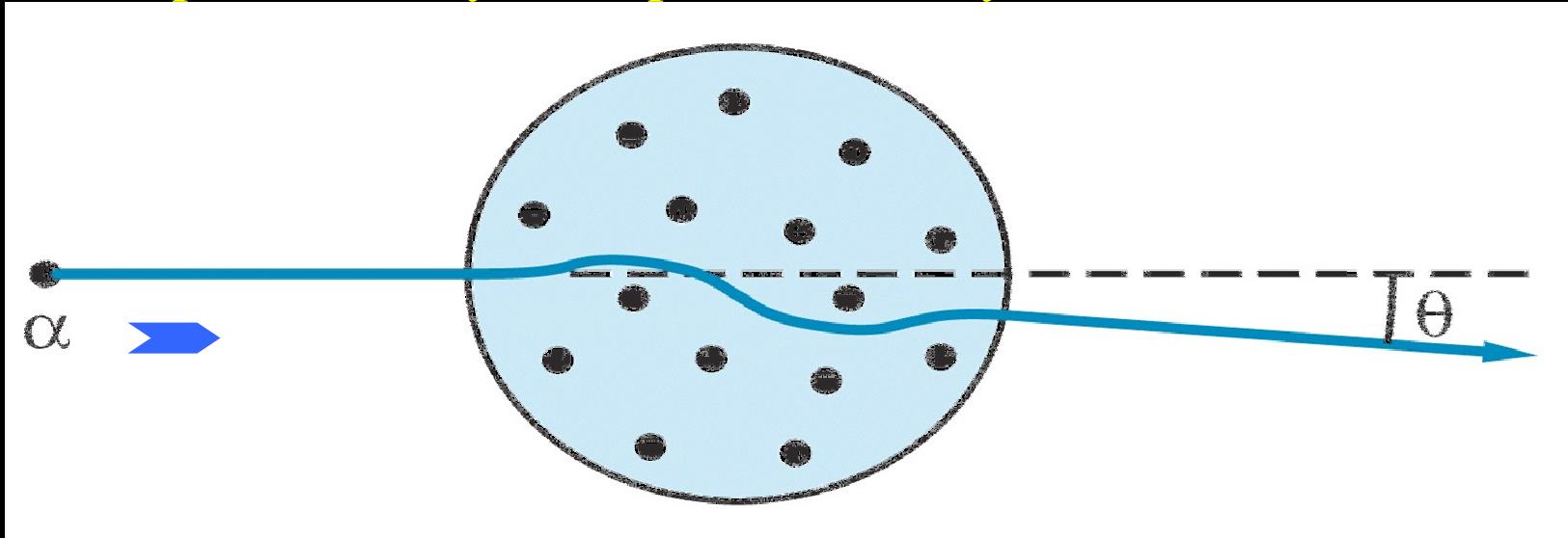
Early Thought: “Plum pudding” model  $\rightarrow$  Atom has a homogenous distribution of Positive charge with electrons embedded in them



- How to test these hypotheses?  $\rightarrow$  Shoot “bullets” at the atom and watch their trajectory. What Kind of bullets ?
  - Indestructible charged bullets  $\rightarrow$  Ionized  $\text{He}^{++}$  atom =  $\alpha^{++}$  particles
  - $Q = +2e$ , Mass  $M_\alpha = 4\text{amu} \gg m_e$ ,  $V_\alpha = 2 \times 10^7 \text{ m/s}$  (non-relativistic)  
[charged to probe charge & mass distribution inside atom]

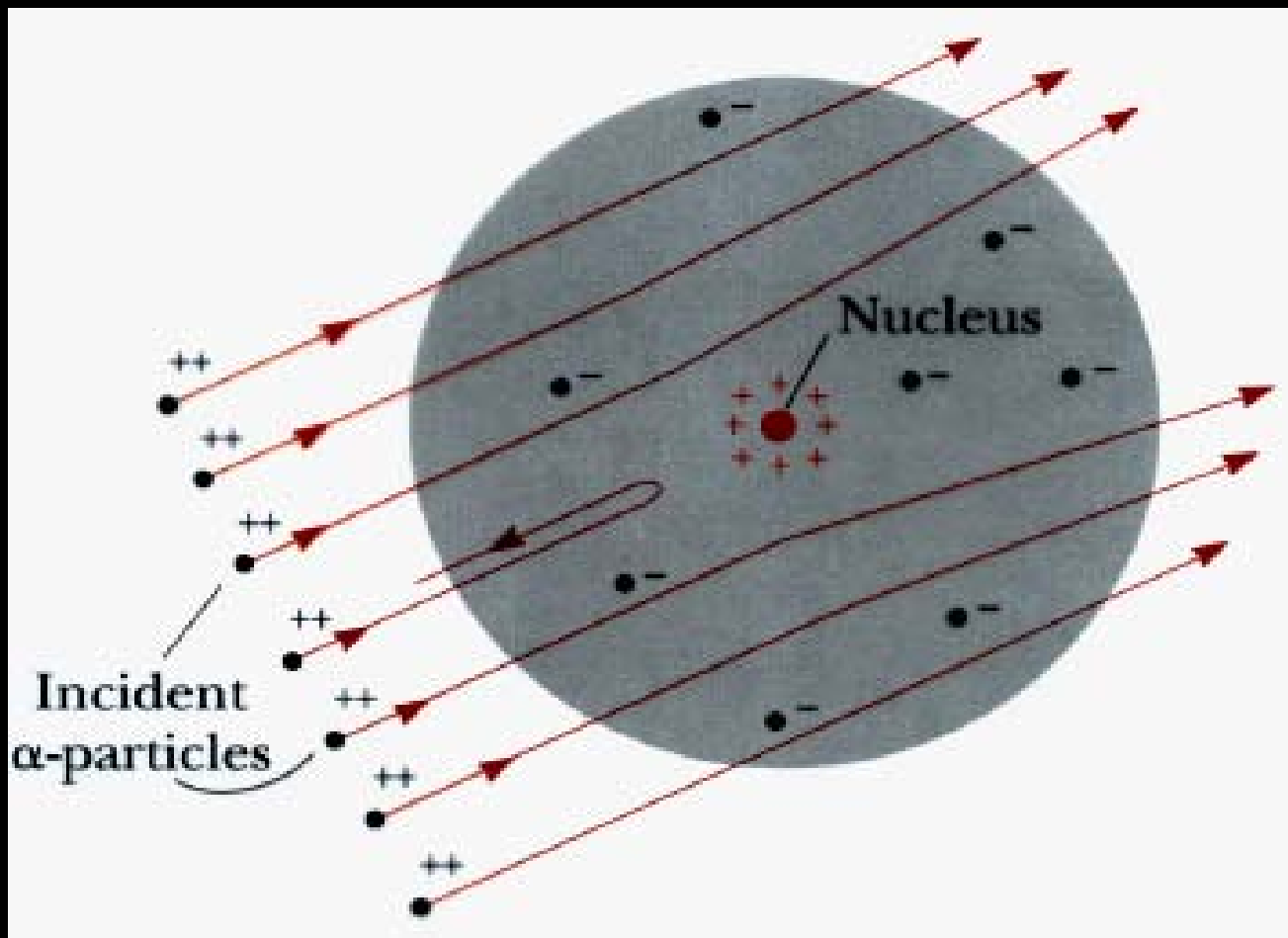
# Plum Pudding Model of Atom

- Non-relativistic mechanics ( $V_{\alpha}/c = 0.1$ )
- In Plum-pudding model,  $\alpha$ -rays hardly scatter because
  - Positive charge distributed over size of atom ( $10^{-10}\text{m}$ )
  - $M_{\alpha} \gg M_e$  (like moving truck hits a bicycle)
  - $\rightarrow$  predict  $\alpha$ -rays will pass thru array of atoms with little scatter

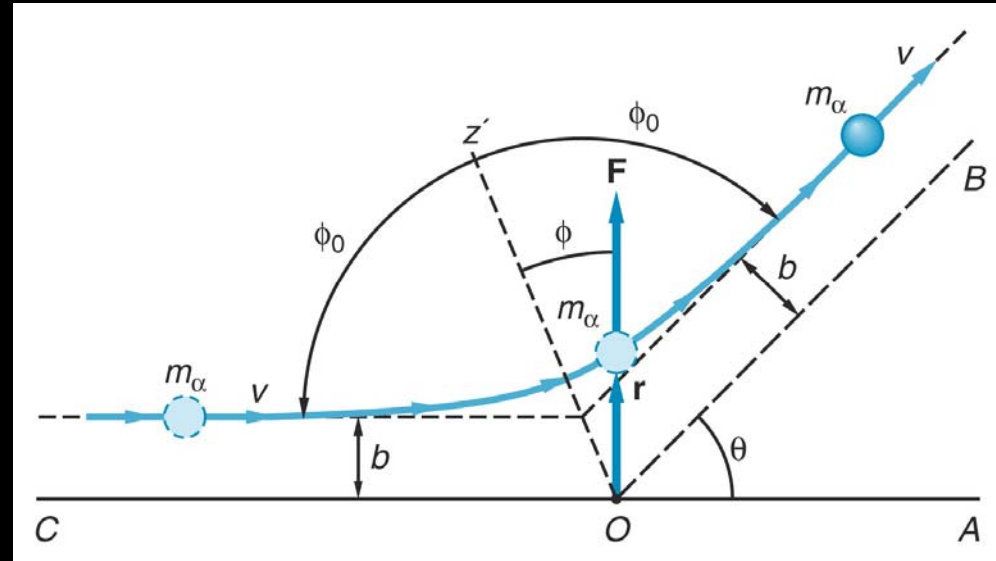
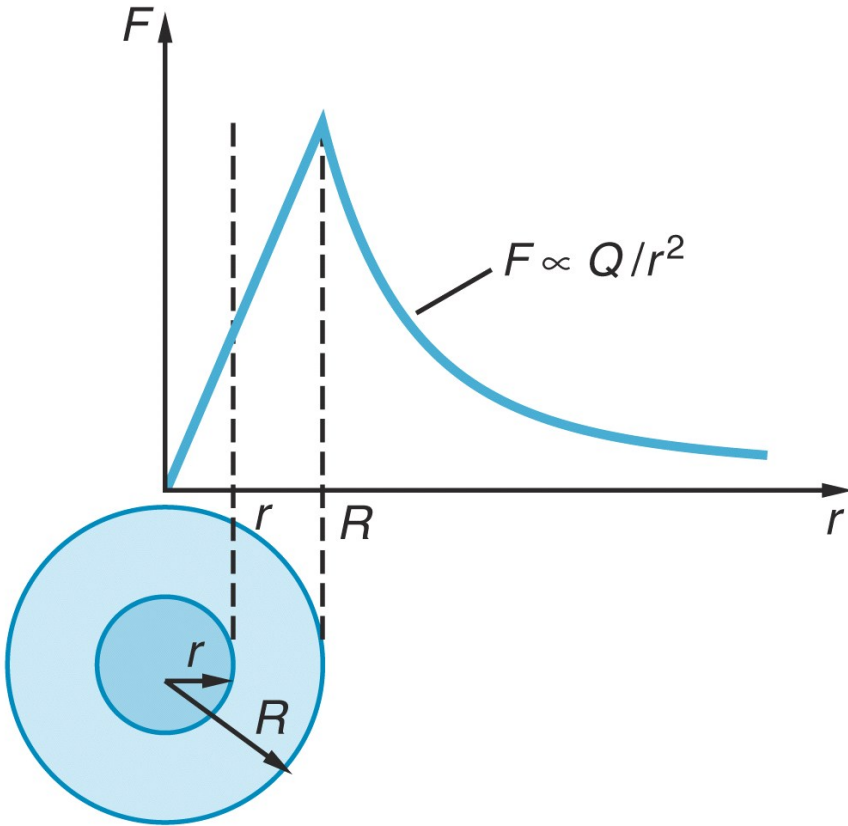


Need to test this hypothesis  $\rightarrow$  Ernest Rutherford

*“Rutherford Scattering” discovered by his PhD Student (Marsden)*



# Force on $\alpha$ -particle due to heavy Nucleus



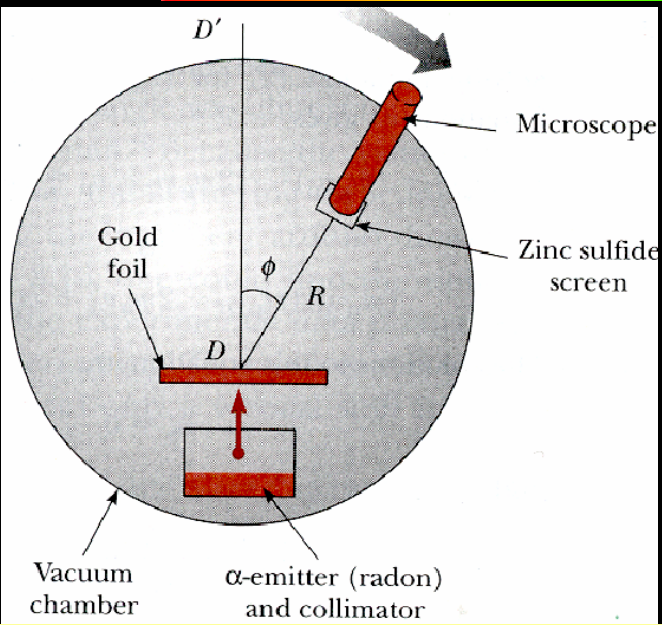
$\alpha$  particle trajectory is hyperbolic

Scattering angle is related to impact par.

$$\text{Impact Parameter } b = \left( \frac{kq_\alpha Q}{m_\alpha v_\alpha^2} \right) \left( \cot \frac{\theta}{2} \right)$$

- Outside radius  $r = R$ ,  $F \propto Q/r^2$
- Inside radius  $r < R$ ,  $F \propto q/r^2 = Qr/R^2$
- Maximum force at radius  $r = R$

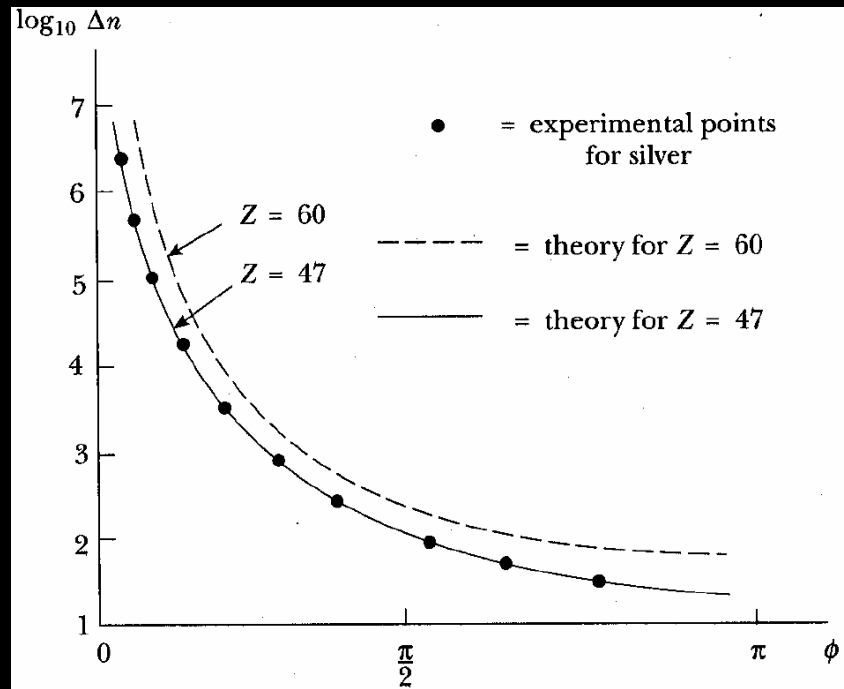
# Rutherford Scattering: Prediction and Experimental Result



$$\Delta n = \frac{k^2 Z^2 e^4 N n A}{4 R^2 \left( \frac{1}{2} m_\alpha v_\alpha^2 \right)^2 \text{Sin}^4(\phi / 2)}$$

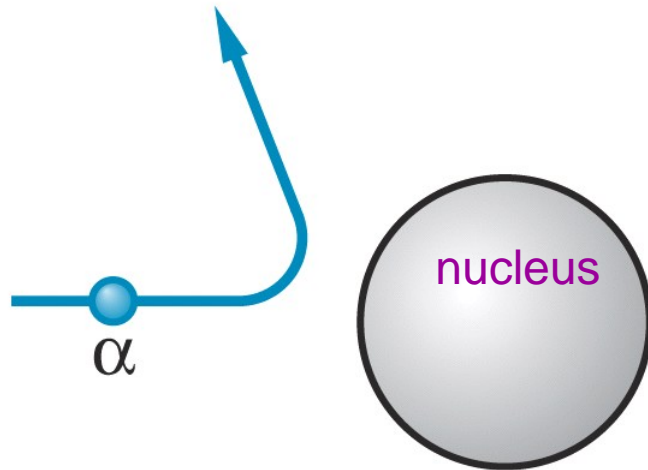
# scattered Vs  $\phi$  depends on :

- $n$  = # of incident alpha particles
- $N$  = # of nuclei/area of foil
- $Ze$  = Nuclear charge
- $K_\alpha$  of incident alpha beam
- $A$  = detector area

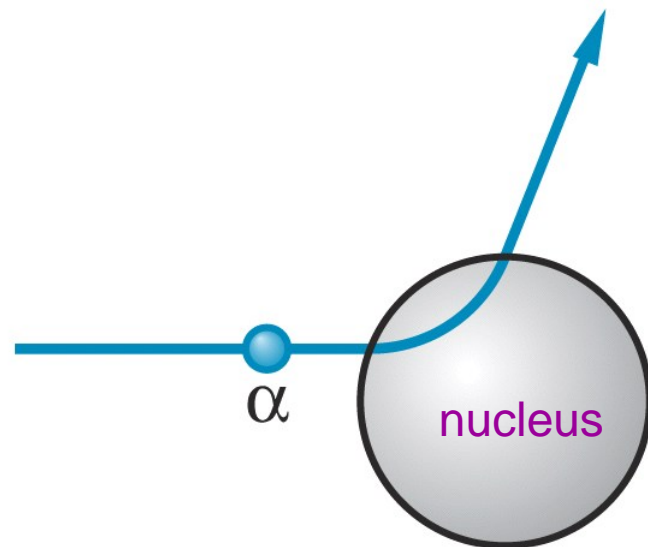


# Rutherford Scattering & Size of Nucleus

(a)



(b)



distance of closest approach  $\propto$  r size of nucleus

$$\text{Kinetic energy of } \alpha = K_{\alpha} = \frac{1}{2} m_{\alpha} v_{\alpha}^2$$

$\alpha$  particle will penetrate thru a radius r until all its kinetic energy is used up to do work AGAINST the Coulomb potential of the Nucleus:

$$K_{\alpha} = \frac{1}{2} m_{\alpha} v_{\alpha}^2 = 8 \text{ MeV} = k \frac{(Ze)(2e)}{r}$$

$$\Rightarrow \boxed{r = \frac{2kZe^2}{K_{\alpha}}}$$

For  $K_{\alpha} = 7.7 \text{ MeV}$ ,  $Z_{\text{Al}} = 13$

$$\Rightarrow r = \frac{2kZe^2}{K_{\alpha}} = 4.9 \times 10^{-15} \text{ m}$$

Size of Nucleus =  $10^{-15} \text{ m}$

Size of Atom =  $10^{-10} \text{ m}$

## *Dimension Matters !*



Size of Nucleus =  $10^{-15} m$

Size of Atom =  $10^{-10} m$

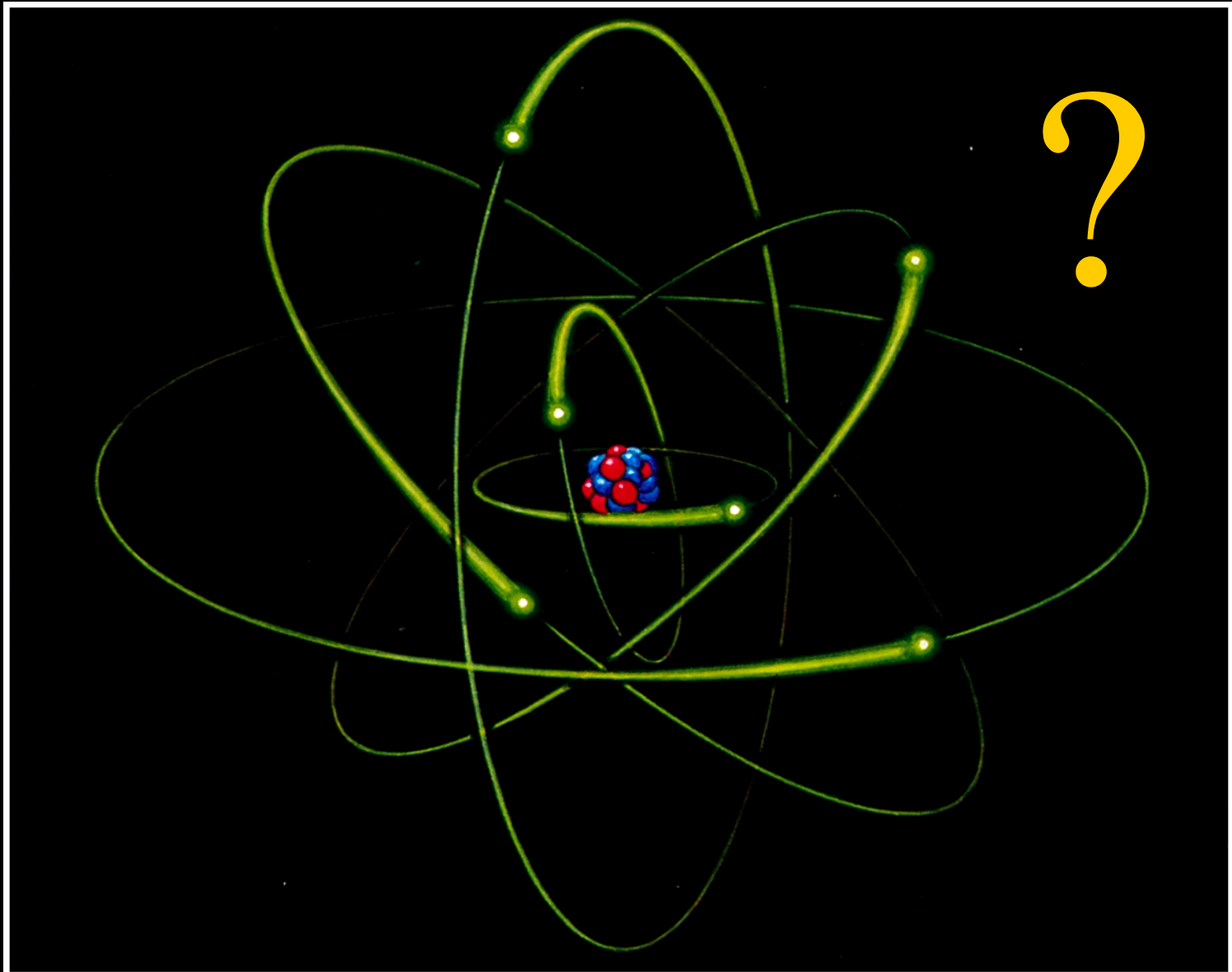
How are the electrons located inside an atom ?

How are they held in a stable fashion ?

necessary condition for us to exist !

All these discoveries will require new experiments and observations

# *Where are the Electrons in an Atom ?*



# Clues: Continuous & Discrete spectra of Elements

Hot blackbody



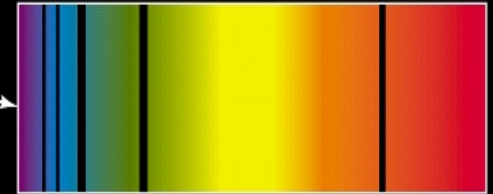
Prism



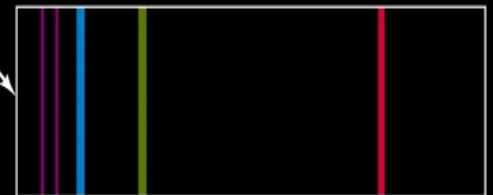
a Continuous spectrum

Cloud of cooler gas

Prism

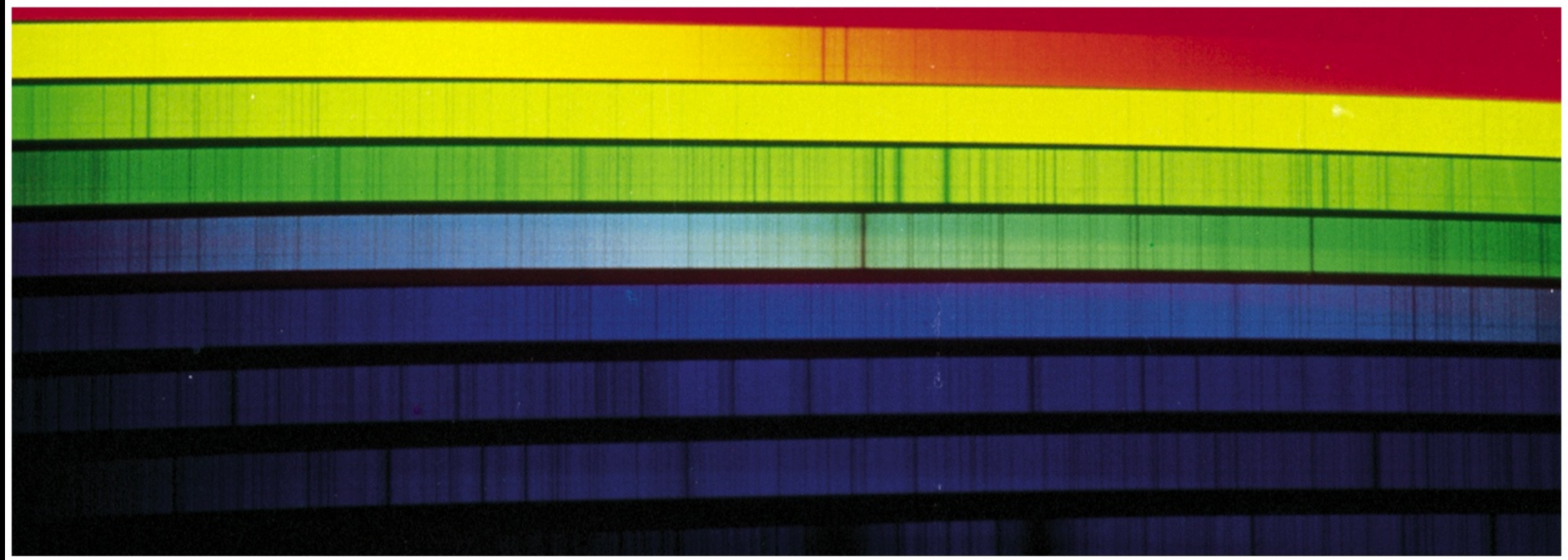


b Absorption line spectrum

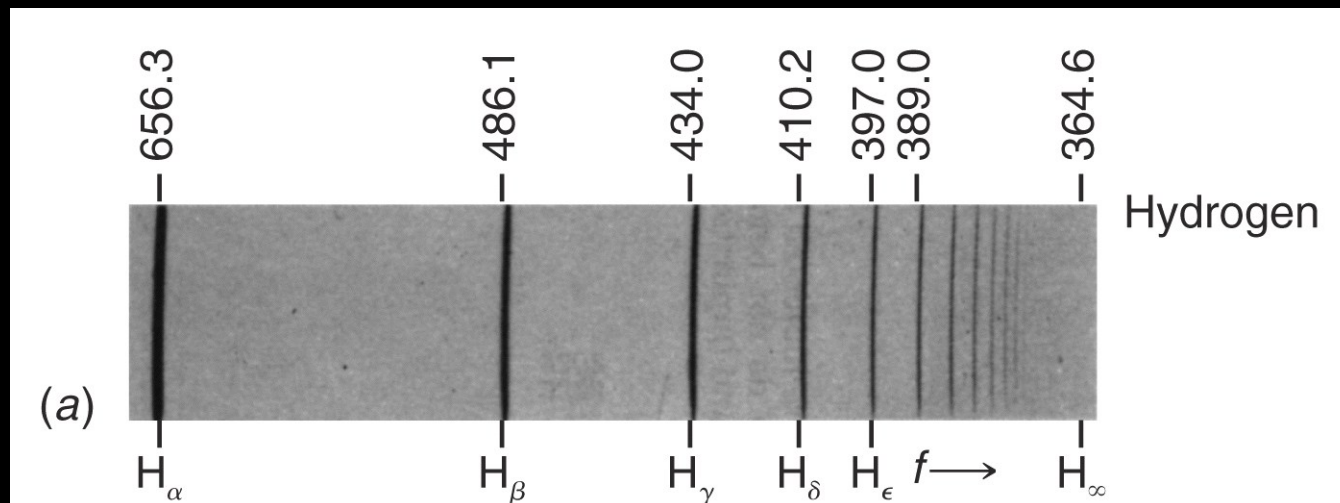
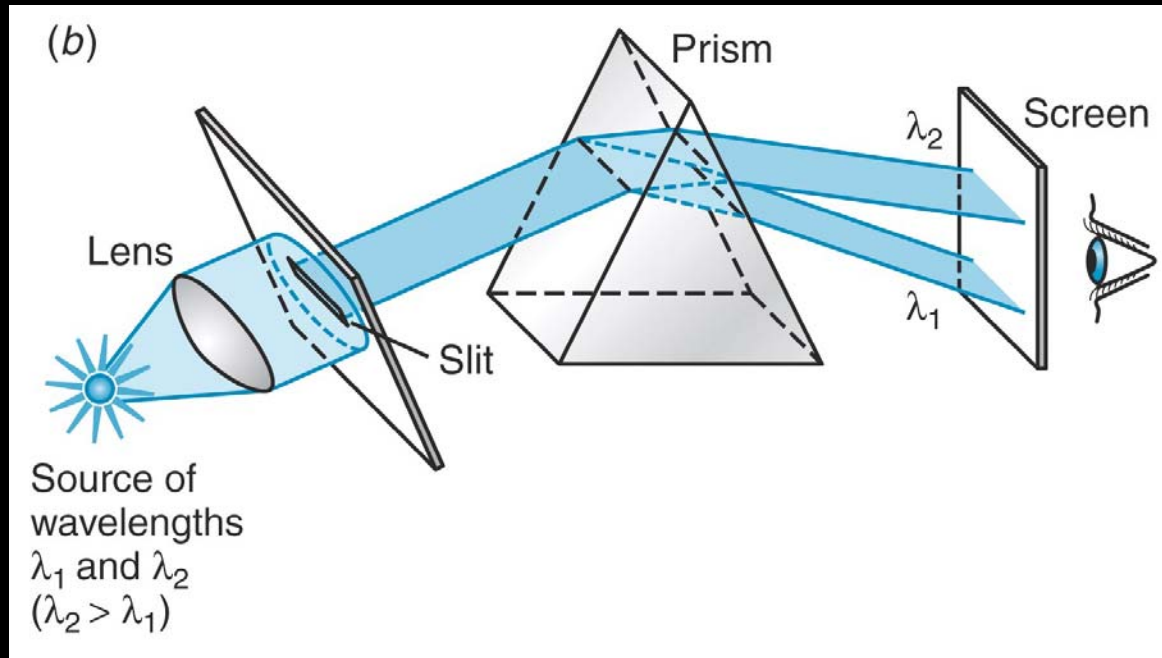


c Emission line spectrum

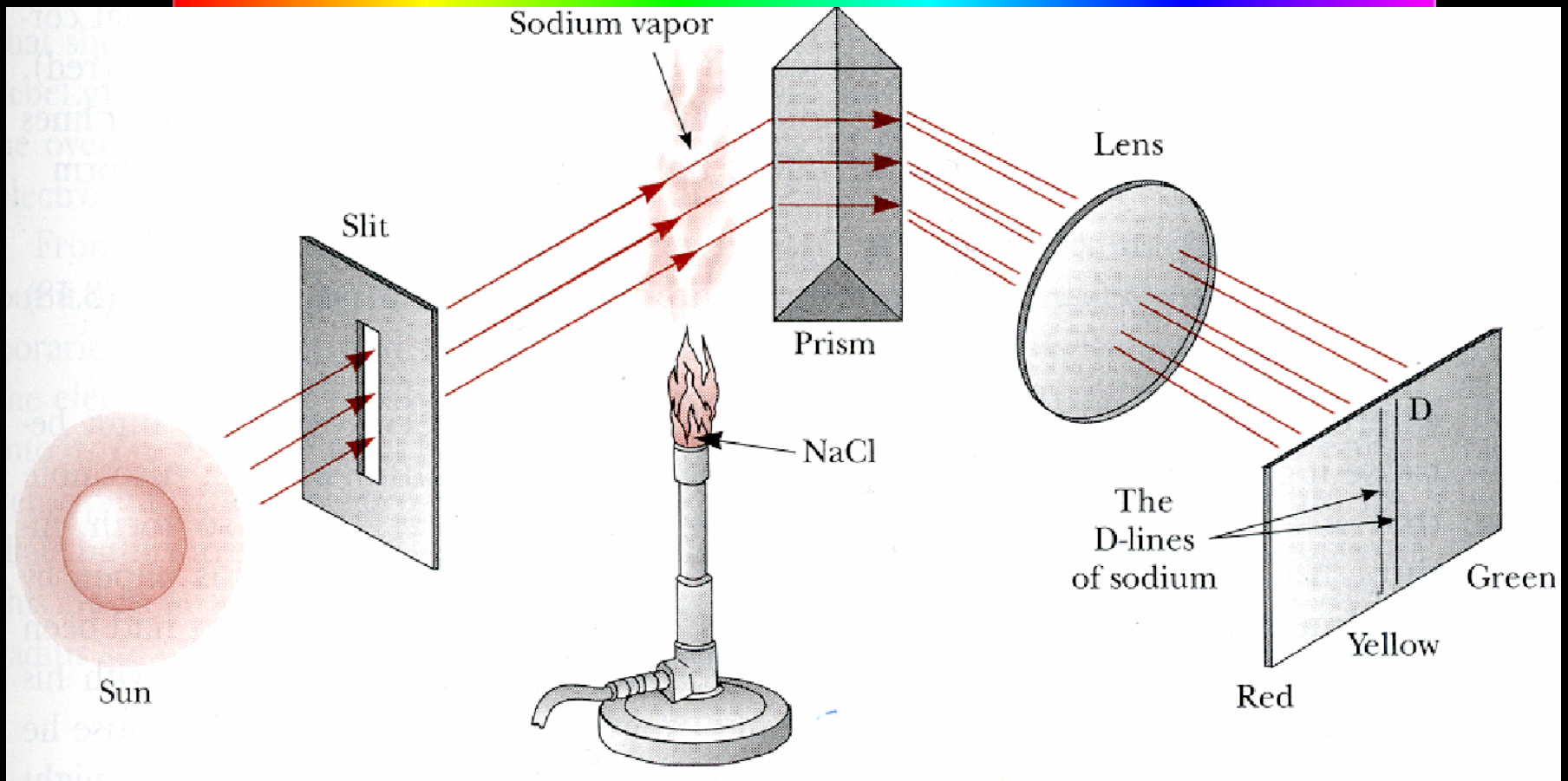
# *Visible Spectrum of Sun Through a Prism*



# Emission & Absorption Line Spectra of Elements



# Kirchhoff's Experiment : "D" Lines in Na

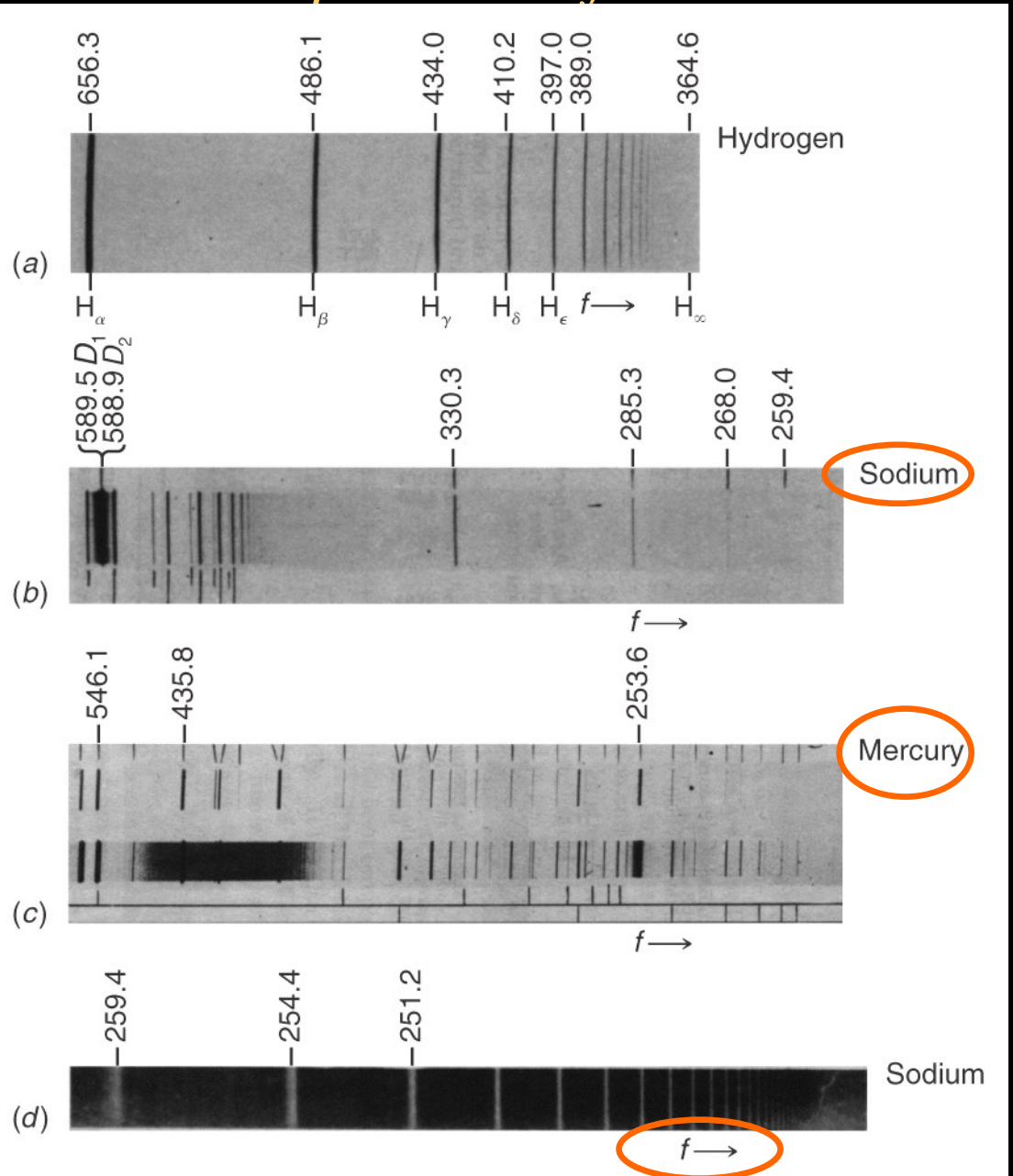


D lines darken noticeably when Sodium vapor introduced  
Between slit and prism

# Emission & Absorption Line Spectrum of Elements



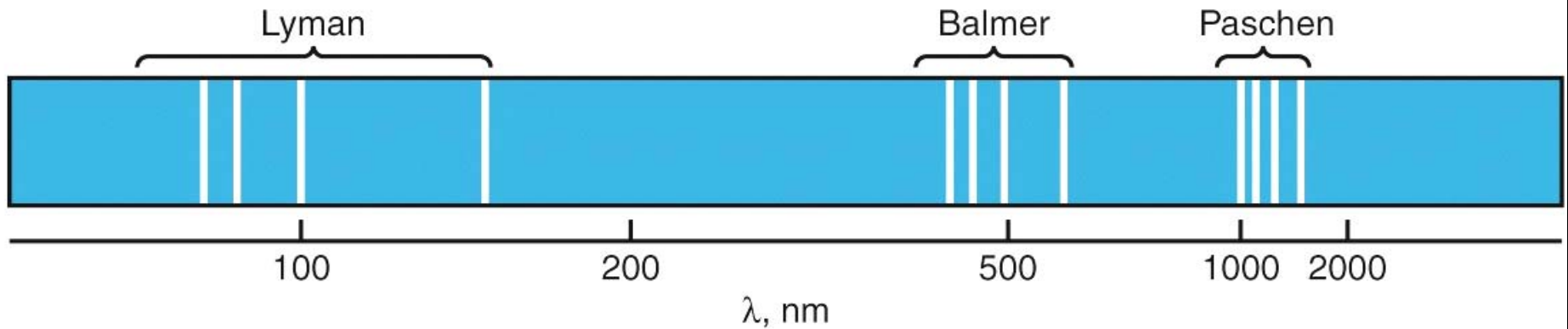
Emission line appear dark because of photographic exposure



Absorption spectrum of Na

While light passed thru Na vapor is absorbed at specific  $\lambda$

# Spectral Observations : series of lines with a pattern



- Empirical observation (by trial & error)
- All these series can be summarized in a simple formula

$$\frac{1}{\lambda} = R \left( \frac{1}{n_f^2} - \frac{1}{n_i^2} \right), n_f > n_i, n_i = 1, 2, 3, 4..$$

Fitting to spectral line series data

$$R = 1.09737 \times 10^7 \text{ m}^{-1}$$

How does one explain this ?

# *The Rapidly Vanishing Atom: A Classical Disaster !*

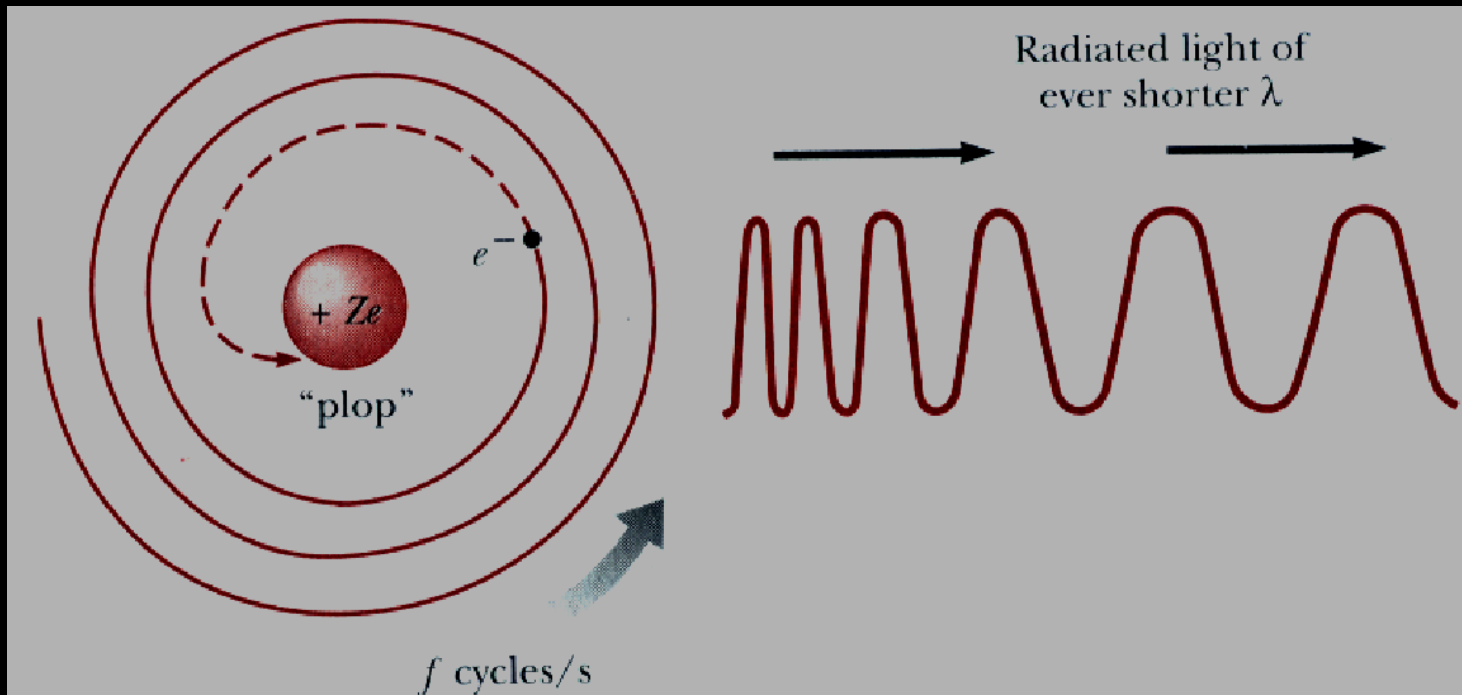
Not too hard to draw analogy with dynamics under another Central Force

Think of the Gravitational Force between two objects and their circular orbits.

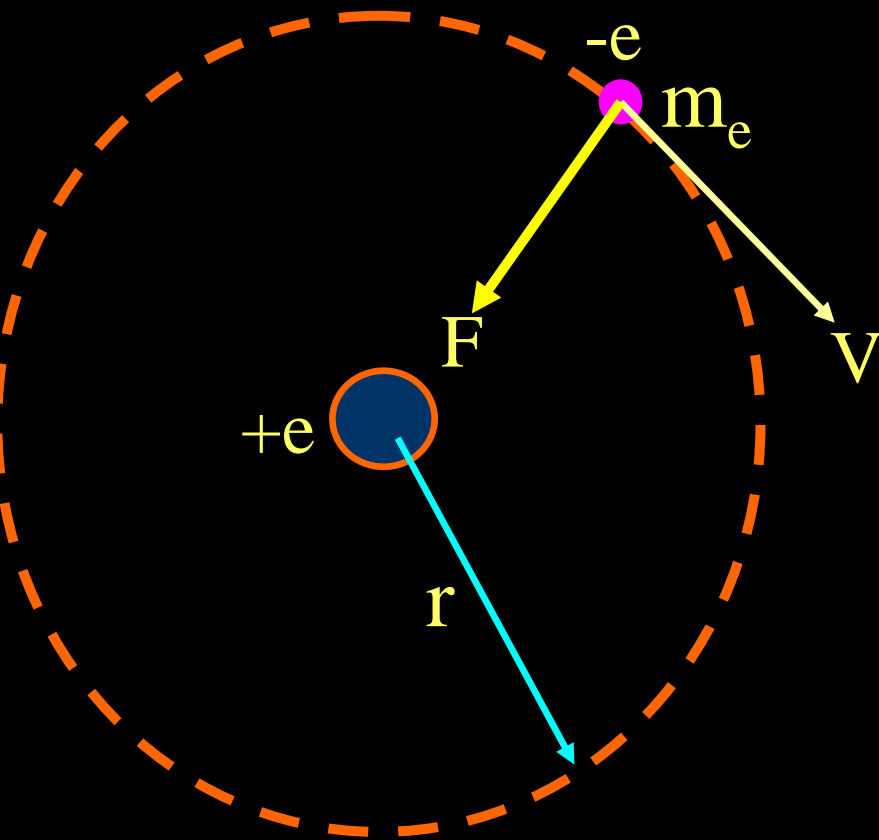
Perhaps the electron rotates around the Nucleus and is bound by their electrical charge

$$F = G \frac{M_1 M_2}{r^2} \Rightarrow k \frac{Q_1 Q_2}{r^2}$$

Laws of E&M destroy this equivalent picture : Why ?



# Bohr's Bold Model of Atom: Semi Quantum/Classical



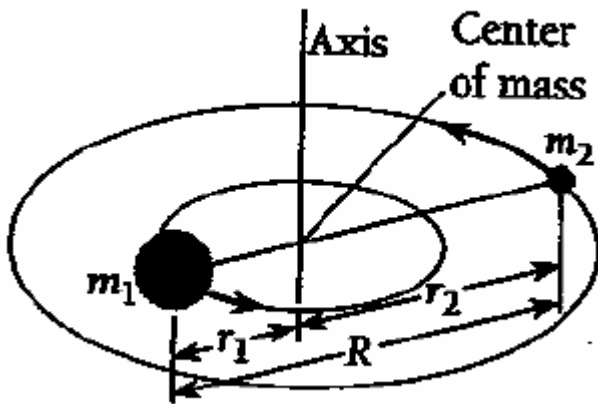
$$U(r) = -k \frac{e^2}{r}$$

$$KE = \frac{1}{2} m_e v^2$$

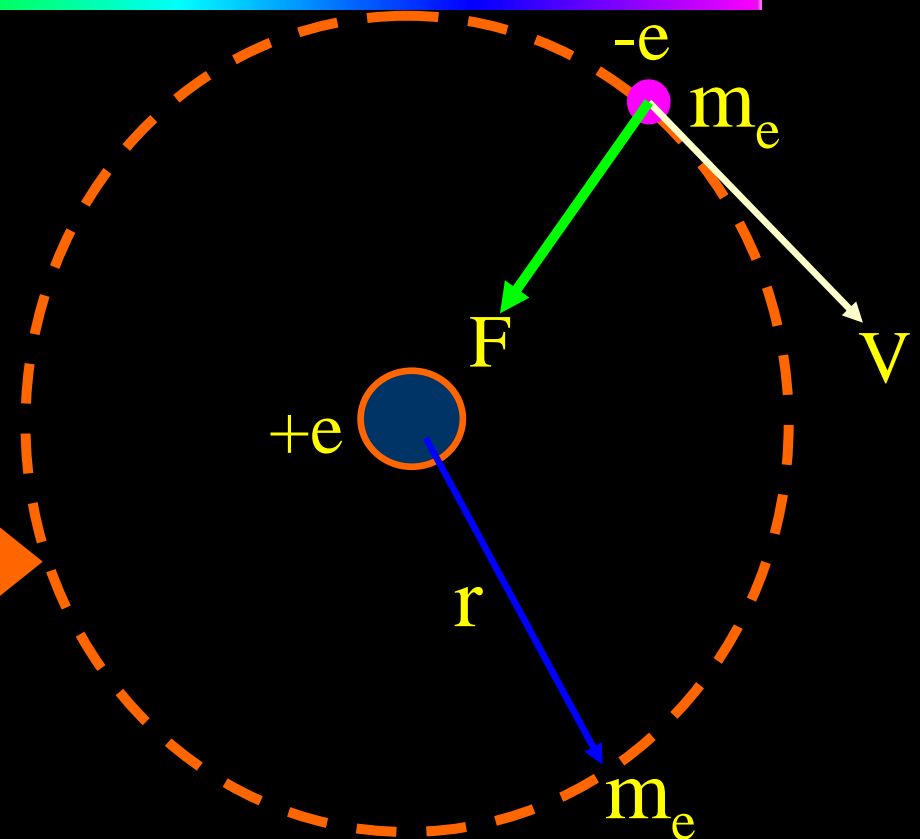
1. Electron in circular orbit around proton with  $vel=v$
2. Only stationary orbits allowed . Electron does not radiate when in these stable (stationary) orbits
3. Orbits quantized:
  - $M_e v r = n h/2\pi$  ( $n=1,2,3\dots$ )
4. Radiation emitted when electron “jumps” from a stable orbit of higher energy  $\rightarrow$  stable orbit of lower energy  $E_f - E_i = hf = hc/\lambda$
5. Energy change quantized
  - $f$  = frequency of radiation

# Reduced Mass of 2-body system

General two body motion under a central force



reduces to



- Both Nucleus &  $e^-$  revolve around their common center of mass (CM)
- Such a system is equivalent to single particle of “reduced mass”  $\mu$  that revolves around position of Nucleus at a distance of ( $e^-$ -N) separation
  - $\mu = (m_e M) / (m_e + M)$ , when  $M \gg m_e$ ,  $\mu = m_e$  (Hydrogen atom)
  - Not so when calculating Muon ( $m_\mu = 207 m_e$ ) or equal mass charges rotating around each other (similar to what you saw in gravitation)

# Allowed Energy Levels & Orbit Radii in Bohr Model

$$E = KE + U = \frac{1}{2} m_e v^2 - k \frac{e^2}{r}$$

Force Equality for Stable Orbit

$\Rightarrow$  Coulomb attraction = CP Force

$$k \frac{e^2}{r^2} = \frac{m_e v^2}{r}$$

$$\Rightarrow KE = \frac{m_e v^2}{2} = k \frac{e^2}{2r}$$

Total Energy  $E = KE + U = -k \frac{e^2}{2r}$

Negative E  $\Rightarrow$  Bound system

This much energy must be added to the system to break up the bound atom

Radius of Electron Orbit :

$$mvr = n\hbar$$

$$\Rightarrow v = \frac{n\hbar}{mr}$$

substitute in KE =  $\frac{1}{2} m_e v^2 = \frac{ke^2}{2r}$

$$\Rightarrow r_n = \frac{n^2 \hbar^2}{mke^2}, n = 1, 2, \dots, \infty$$

$n = 1 \Rightarrow$  Bohr Radius  $a_0$

$$a_0 = \frac{1^2 \hbar^2}{mke^2} = 0.529 \times 10^{-10} m$$

In general  $r_n = n^2 a_0; n = 1, 2, \dots, \infty$

Quantized orbits of rotation

# Energy Level Diagram and Atomic Transitions

$$E_n = K + U = \frac{-ke^2}{2r}$$

since  $r_n = a_0 n^2$ ,  $n = \text{quantum number}$

$$E_n = \frac{-ke^2}{2a_0 n^2} = -\frac{13.6}{n^2} \text{ eV}, \quad n = 1, 2, 3, \dots, \infty$$

Interstate transition:  $n_i \rightarrow n_f$

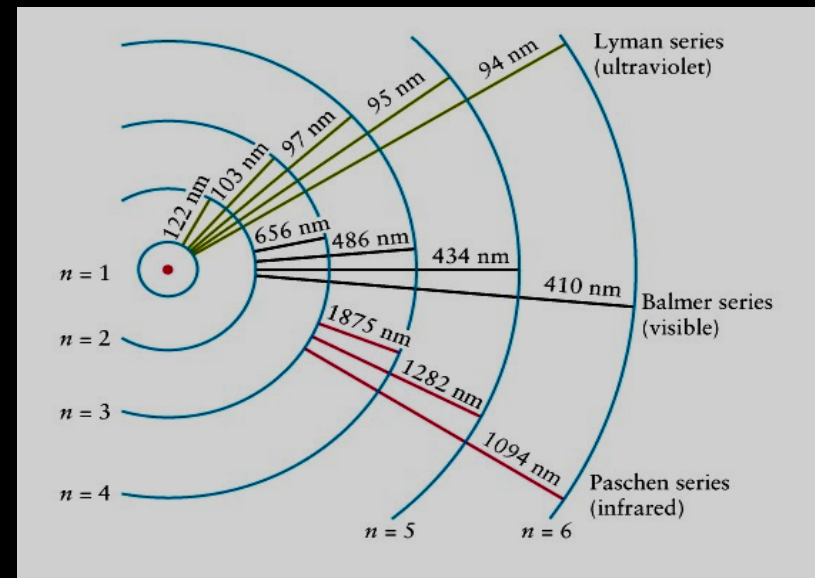
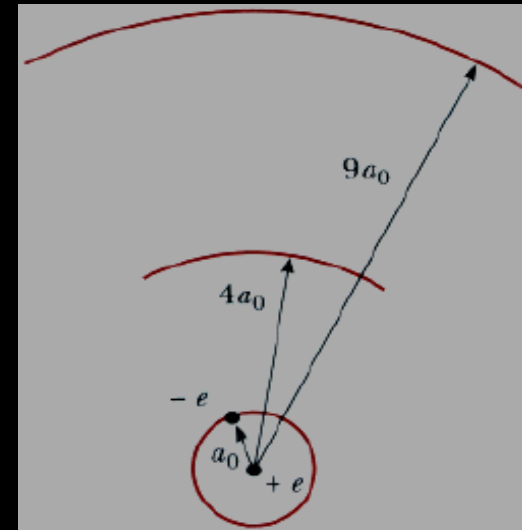
$$\Delta E = hf = E_i - E_f$$

$$= \frac{-ke^2}{2a_0} \left( \frac{1}{n_i^2} - \frac{1}{n_f^2} \right)$$

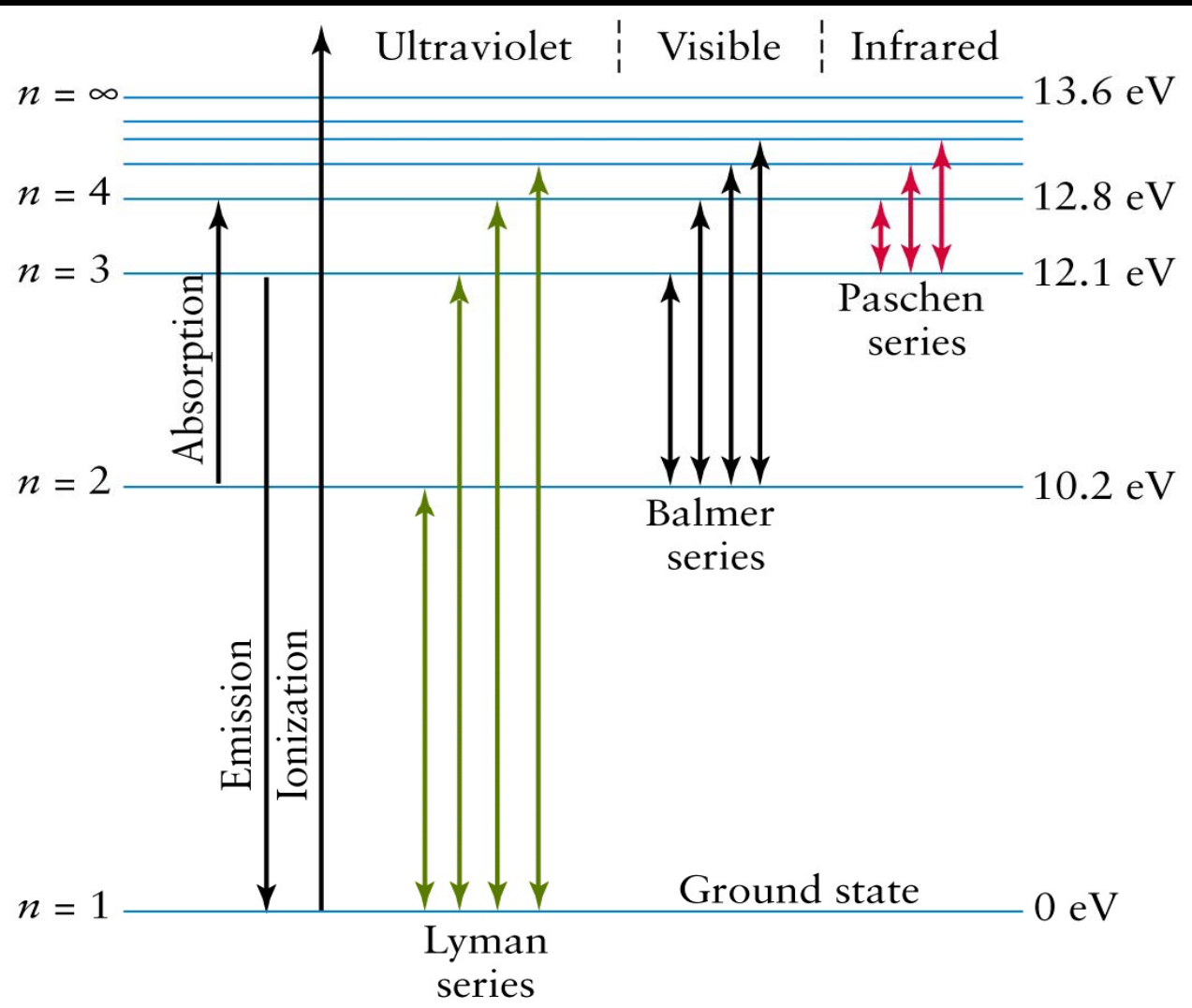
$$f = \frac{ke^2}{2ha_0} \left( \frac{1}{n_f^2} - \frac{1}{n_i^2} \right)$$

$$\frac{1}{\lambda} = \frac{f}{c} = \frac{ke^2}{2hca_0} \left( \frac{1}{n_f^2} - \frac{1}{n_i^2} \right)$$

$$= \mathbf{R} \left( \frac{1}{n_f^2} - \frac{1}{n_i^2} \right)$$



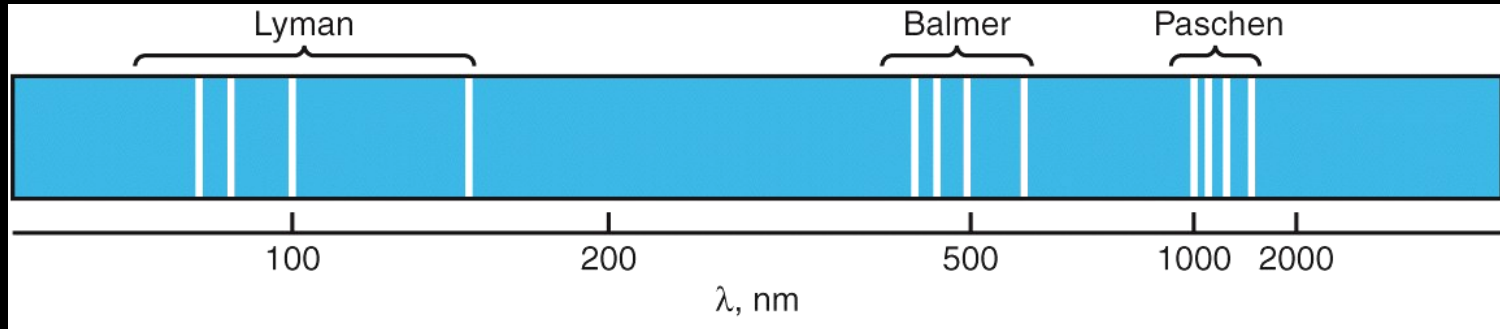
# Hydrogen Spectrum: as explained by Bohr



$$E_n = - \left( \frac{ke^2}{2a_0} \right) \frac{Z^2}{n^2}$$

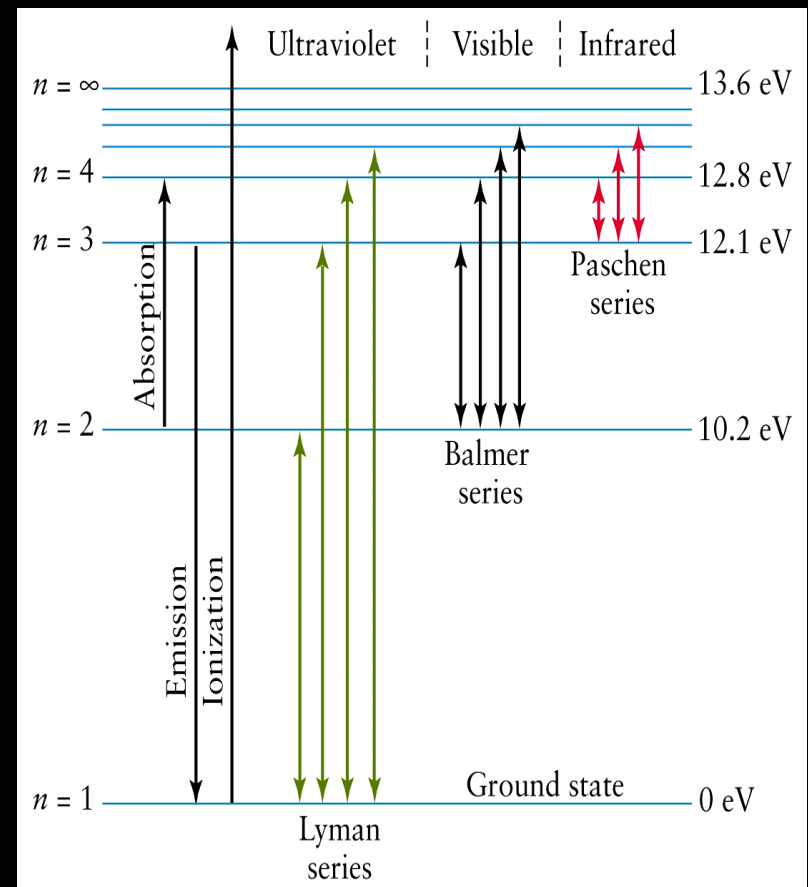
Bohr's "R" same as Rydberg Constant **R** derived empirically from spectral series

# A Look Back at the Spectral Lines With Bohr's Optic

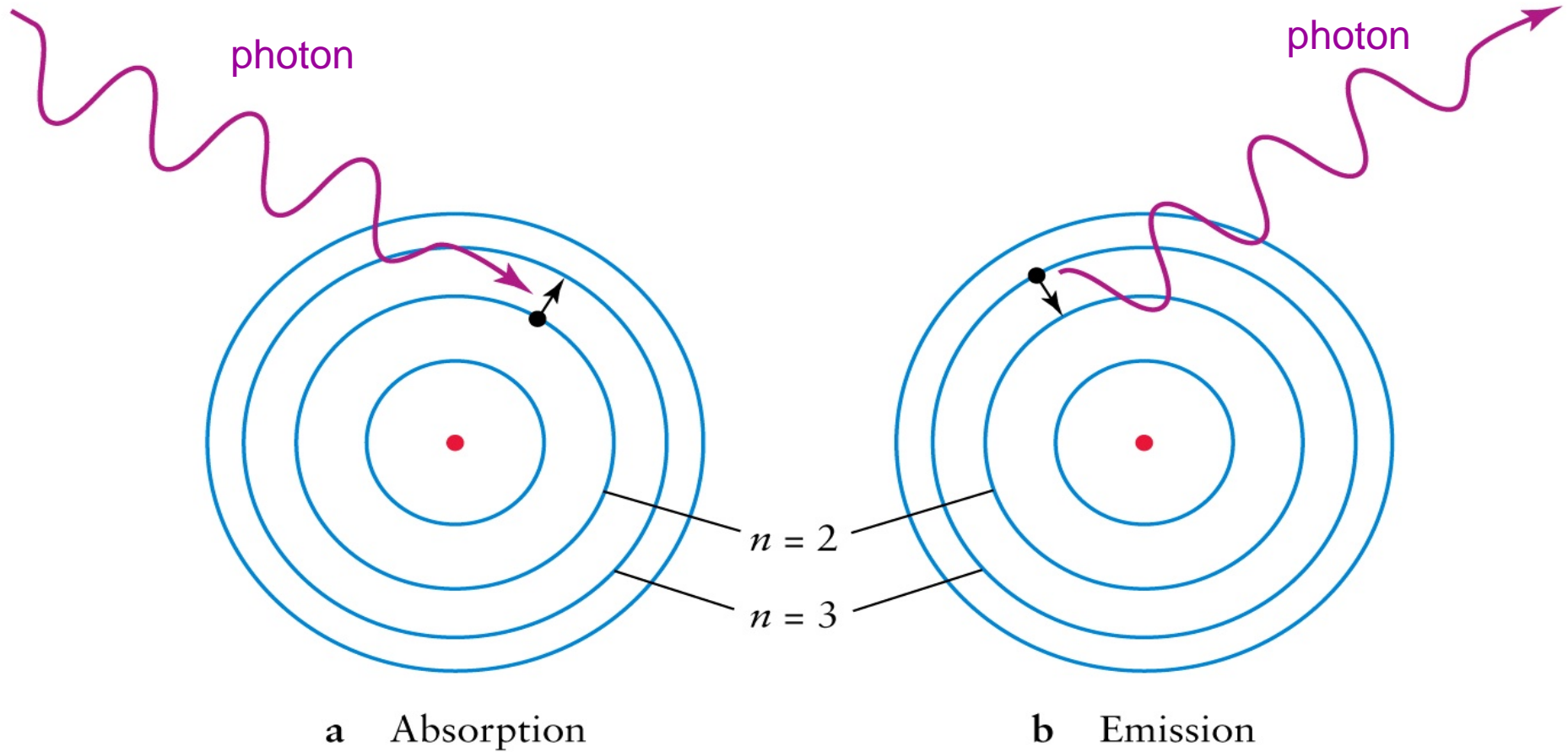


$$E_n = - \left( \frac{ke^2}{2a_0} \right) \frac{Z^2}{n^2}$$

**Rydberg Constant**



# Bohr's Atom: Emission & Absorption Spectra



## Some Notes About Bohr Like Atoms

- Ground state of Hydrogen atom (n=1)  $E_0 = -13.6 \text{ eV}$
- Method for calculating energy levels etc applies to all Hydrogen-like atoms  $\rightarrow -1e$  around  $+Ze$ 
  - Examples :  $\text{He}^+$ ,  $\text{Li}^{++}$
- Energy levels would be different if replace electron with Muons
- Bohr's method can be applied in general to all systems under a central force (e.g. gravitational instead of Coulombic)

If change  $U(r) = k \frac{Q_1 Q_2}{r} \rightarrow G \frac{M_1 M_2}{r}$

Changes every thing: E, r , f etc

"Importance of constants in your life"