4E : The Quantum Universe

Lecture 19, May 3

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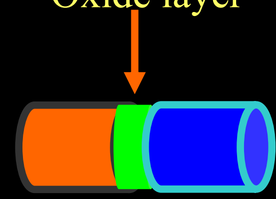


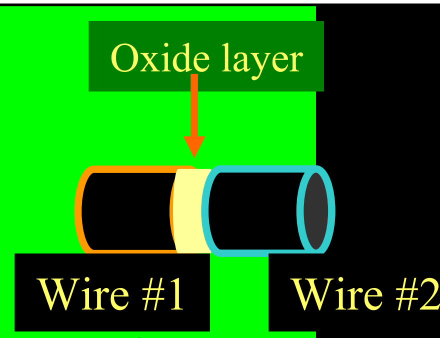
Q: 2 Cu wires are seperated by insulating Oxide layer. Modeling the Oxide layer as a square barrier of height U=10.0eV, estimate the transmission coeff for an incident beam of electrons of E=7.0 eV when the layer thickness is
(a) 5.0 nm (b) 1.0nm

Q: If a 1.0 mA current in one of the intwined wires is incident on Oxide layer, how much of this current passes thru the Oxide layer on to the adjacent wire if the layer thickness is 1.0nm? What becomes of the remaining current?

$$\mathrm{T(E)} = \left[1+\frac{1}{4}\left(\frac{U^2}{E(U-E)}\right)\sinh^2(\alpha L)\right]^{-1}$$

$$\alpha = \frac{\sqrt{2\mathrm{m(U\text{-}E)}}}{\hbar}, k = \frac{\sqrt{2\mathrm{mE}}}{\hbar}$$

$$T(E) = \left[1+\frac{1}{4}\left(\frac{U^2}{E(U-E)}\right)\sinh^2(\alpha L)\right]^{-1}$$



Use $\hbar$=1.973 keV.Å/c , m_e = 511 keV/c^2

$$\Rightarrow \alpha = \frac{\sqrt{2m_e(U-E)}}{\hbar} = \frac{\sqrt{2\times 511 kev/c^2 (3.0\times 10^{-3} keV)}}{1.973 \text{ keV.Å/c}} = 0.8875\text{Å}^{-1}$$

Substitute in expression for T=T(E)

$$T= \left[1+\frac{1}{4}\left(\frac{10^2}{7(10-7)}\right)\sinh^2(0.8875\text{Å}^{-1})(50\text{Å})\right]^{-1} = 0.963\times 10^{-38}(small)!!$$

However, for L=10Å; T=0.657×10^{-7}

Reducing barrier width by $\times 5$ leads to Trans. Coeff enhancement by 31 orders of magnitude !!!

1 mA current =I=$\frac{\text{Q=Nq}_e}{\text{t}}$ $\Rightarrow N=6.25\times 10^{15}$ *electrons*

N_T=# of electrons that escape to the adjacent wire (past oxide layer)

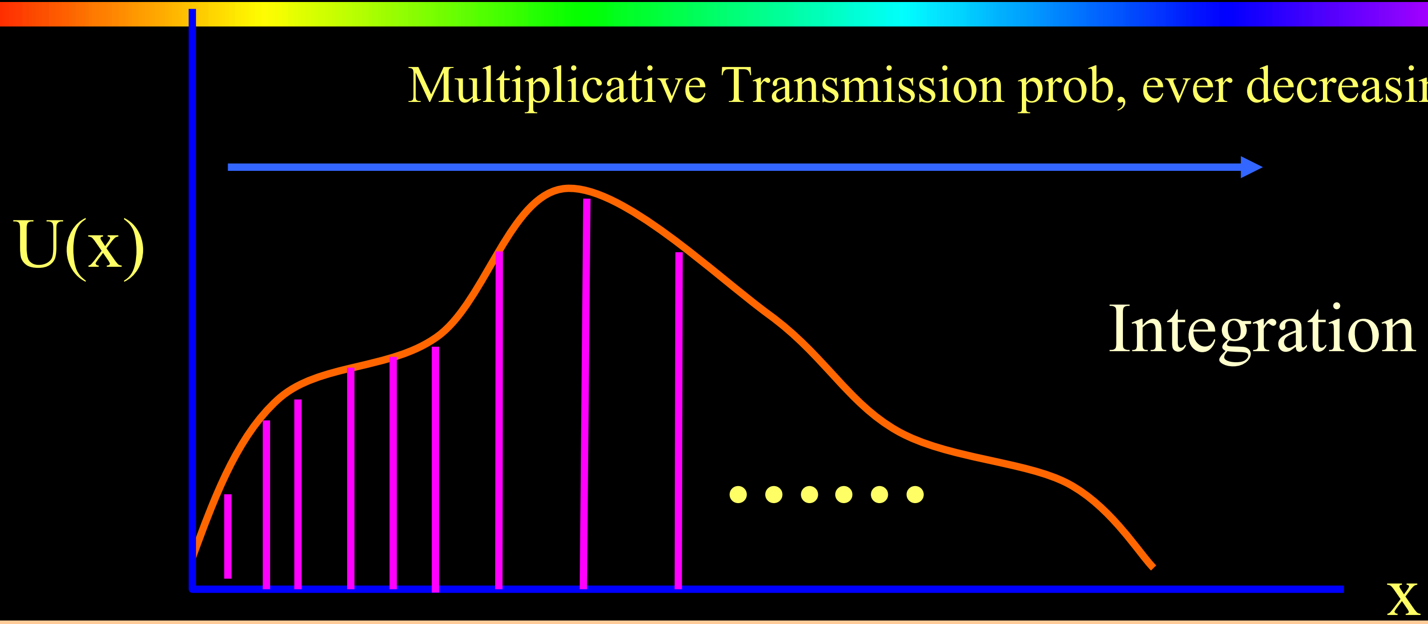
$N_T = N.T = (6.25\times 10^{15}\ electrons)\times \boxed{T}$;

For L=10 Å, T=0.657×10^{-7} $\Rightarrow N_T = 4.11\times 10 \Rightarrow \boxed{I_T = 65.7\,pA}$!!

Current Measured on the first wire is sum of incident+reflected currents and current measured on "adjacent" wire is the I_T

Oxide thickness makes all the difference !
That's why from time-to-time one needs to
Scrape off the green stuff off the naked wires

A Complicated Potential Barrier Can Be Broken Down



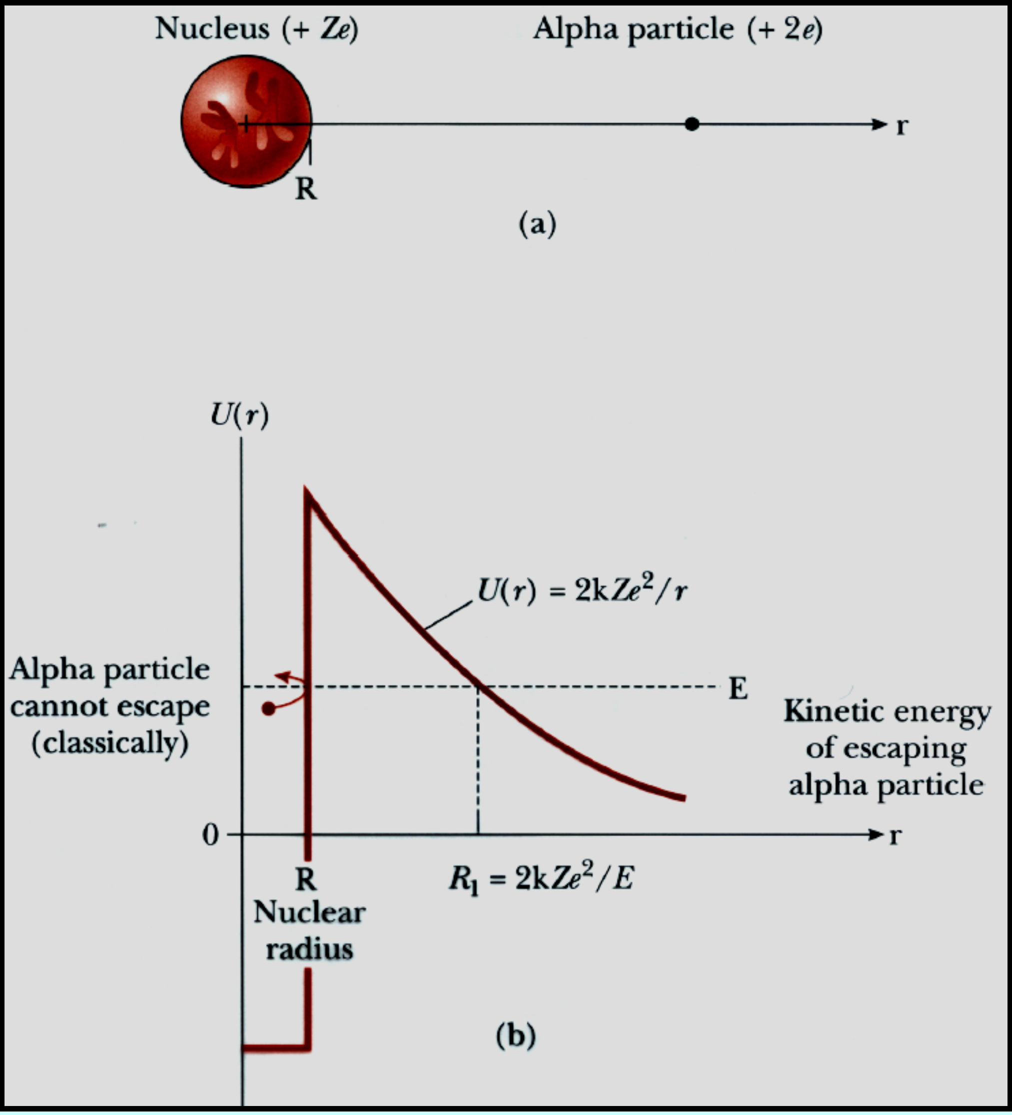
Can be broken down into a sum of successive Rectangular potential barriers for which we learnt to find the Transmission probability T_i

The Transmitted beam intensity thru one small barrier becomes incident beam intensity for the following one

So on & so forth …till the particle escapes with final Transmission prob T

$$T = \int T_i \, dx = e^{-2\left[\frac{\sqrt{2m}}{\hbar}\int\sqrt{U(x)-E}\,dx\right]}$$

Radioactivity: The α-particle & Steve McQueen Compared



$$T = \int T_i \; dx = e^{-2\left[\frac{\sqrt{2m_\alpha}}{\hbar}\int\sqrt{\frac{2ke^2Z}{r} - E_\alpha}\,dx\right]}$$

- In a Nucleus such as Ra, Uranium etc α particle rattles around parent nucleus, “hitting”the nuclear walls with a very high frequency (probing the “fence”), if the Transmission prob T>0, then eventually particle escapes
- Within nucleus, α particle is virtually free but is trapped by the Strong nuclear force
- Once outside nucleus, the particle “sees” only the repulsive (+) columbic force (nuclear force too faint outside) which keeps it within nucleus
- Nuclear radius R = 10^{-14} m, E_α = 9MeV
- Coulomb barrier U(r) =kq_1q_2/r
  - At r=R, U(R) ≈ 30 MeV barrier
- α-particle, due to QM, tunnels thru
- It’s the sensitivity of T on E_α that accounts for the wide range in half-lives of radioactive nuclei

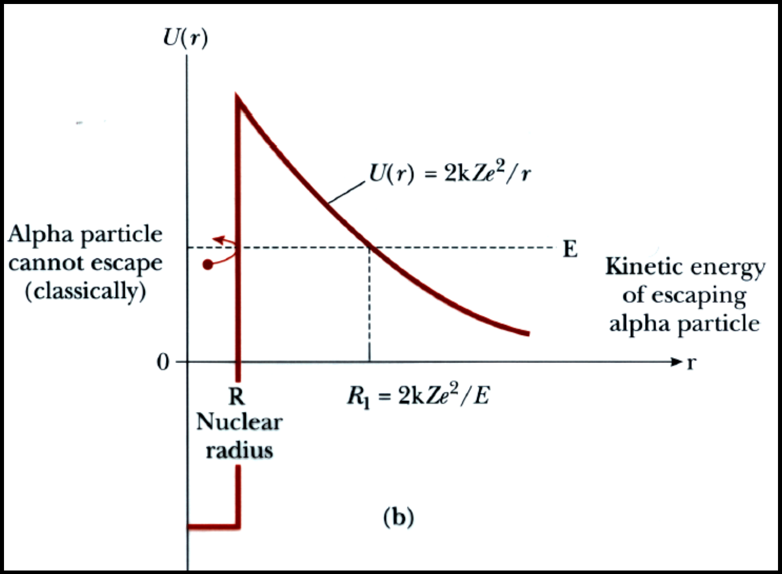
Radioactivity Explained Roughly (..is enough!)

- Protons and neutrons rattling freely inside radioactive nucleus ($R \cong 10^{-15}$m)
- Constantly morphing into clusters of protons and neutrons
- Proto-alpha particle =(2p+2n) of $\approx$ 9 MeV prevented from getting out by the imposing Coulombic repulsion of remaining charge ($\approx$ 30MeV)
- Escapes by tunneling thru Coloumb potential…but some puzzling features:
- α particles emitted from all types of radioactive nuclei have roughly same KE $\cong$ 4-9 MeV

$$T = \int T_i \, dx = e^{-2\left[\frac{\sqrt{2m_\alpha}}{\hbar}\int\sqrt{\frac{2ke^2Z}{r} - E_\alpha}\,dx\right]}$$

- In contrast, the half live T(N→ e^{-1} N) differ by more than 20 orders of magnitude !

Element	KE of emitted α	Half Life
^{212}Po	8.95 MeV	3×10^{-7} s
^{240}Cm	6.40 MeV	27 days
^{226}Ra	4.90 MeV	1.60×10^{3} Yr
^{232}Th	4.05 MeV	1.41×10^{10} yr



Radioactivity Explained Crudely

$$T(E) \approx e^{\left[\frac{-2}{\hbar}\sqrt{2m}\int\sqrt{U(x)-E}\;dx\right]}, U(x) = \frac{2e^2 Z}{4\pi\varepsilon_0 r}$$

$$\ln T = \frac{-2}{\hbar}\int_0^b \sqrt{2m_\alpha\left(\frac{2e^2 Z}{4\pi\varepsilon_0 r} - E_\alpha\right)}\;dr,$$

limits of integration correspond to values of r when $\boxed{E=U}$

$$\Rightarrow \frac{2e^2 Z}{4\pi\varepsilon_0 b} = E \Rightarrow \boxed{b = \frac{2e^2 Z}{4\pi\varepsilon_0 E_\alpha}}$$

Define $\xi = \frac{r}{b}; = \frac{r}{2e^2 Z / 4\pi\varepsilon_0 E} \Rightarrow \ln T \cong \frac{-2\left(\sqrt{2m_\alpha E}\right)b}{\hbar}\int_0^1 \sqrt{\frac{1}{\xi} - 1}\;d\xi$

Substitute $\xi = \sin^2\theta$ in integration, change limits $\Rightarrow$

$$\ln T \cong \frac{-4\left(\sqrt{2m_\alpha E_\alpha}\right)b}{\hbar}\boxed{\int_0^{\pi/2}\cos^2\theta d\theta}; \text{use} \int_0^{\pi/2}\cos^2\theta d\theta = \pi/4 \;\&\; E_\alpha = \frac{m_\alpha V_\alpha^2}{2}$$

$$\ln T \cong \frac{-2\pi}{\hbar}\frac{Ze^2}{4\pi\varepsilon_0}\sqrt{\frac{2m_\alpha}{E_\alpha}} = \frac{-4\pi}{\hbar}\frac{Ze^2}{4\pi\varepsilon_0}\frac{1}{V_\alpha} \Rightarrow \boxed{T \cong e^{\left[\frac{-4\pi}{\hbar}\frac{Ze^2}{4\pi\varepsilon_0}\frac{1}{V_\alpha}\right]}}$$

$\Rightarrow \boxed{T \propto e^{\frac{-1}{V_\alpha}} \text{ and } T \propto e^{-Z}}$ ....SHARP DEPENDENCE!!

Radioactivity

A more eloberate calculation (Bohm) yields $T(E)=e^{\left\{-4\pi Z\sqrt{\frac{E_0}{E}}+8\sqrt{\frac{ZR}{r_0}}\right\}}$

where $r_0 = \frac{\hbar^2}{m_\alpha ke^2} \simeq 8\text{fm}$ is the "Bohr Radius" of alpha particles

and $E_0 = \frac{ke^2}{2r_0} = 0.0993 MeV$ = Nuclear "Rydberg"

To obtain decay rates, need to multiply T(E) by the number of collisions

α particle makes with the "walls" of the nuclear barrier. This collission frequency

$f = \frac{V_\alpha}{2R}$ = transit time for α particle crossing the nuclear barrier (rattle time)

Typically $f = 10^{21}$ collissions/second

Decay rate (prob. of α emission per unit time) $\lambda = f \bullet T(E)$

$$\lambda = 10^{21} e^{\left\{-4\pi Z\sqrt{\frac{E_0}{E_\alpha}}+8\sqrt{\frac{ZR}{r_0}}\right\}}$$

Definition : Half life $t_{1/2} = \frac{\ln 2}{\lambda}$

Half Lives Compared: Sharp dependence on E_α

α particles emerge with (a) E=4.05 MeV in Thorium (b) E=8.95 MeV in Polonium.

The Nuclear size R = 9 fm in both cases. Which one will outlive you ?

Thorium (Z=90) decays into Radium (Z=88)

$$T(E) = \exp\left\{-4\pi(\boxed{88})\sqrt{(0.0993/4.05)} + 8\sqrt{\boxed{88}(9.00/7.25)}\right\}$$

$$=1.3\times10^{-39}$$

Taking f=$10^{21}\,Hz \Rightarrow \lambda = 1.3\times10^{-18}$ α emission $\Rightarrow t_{1/2} = \dfrac{0.693}{1.3\times10^{-18}} = 1.7\times10^{10}\,yr$!!!

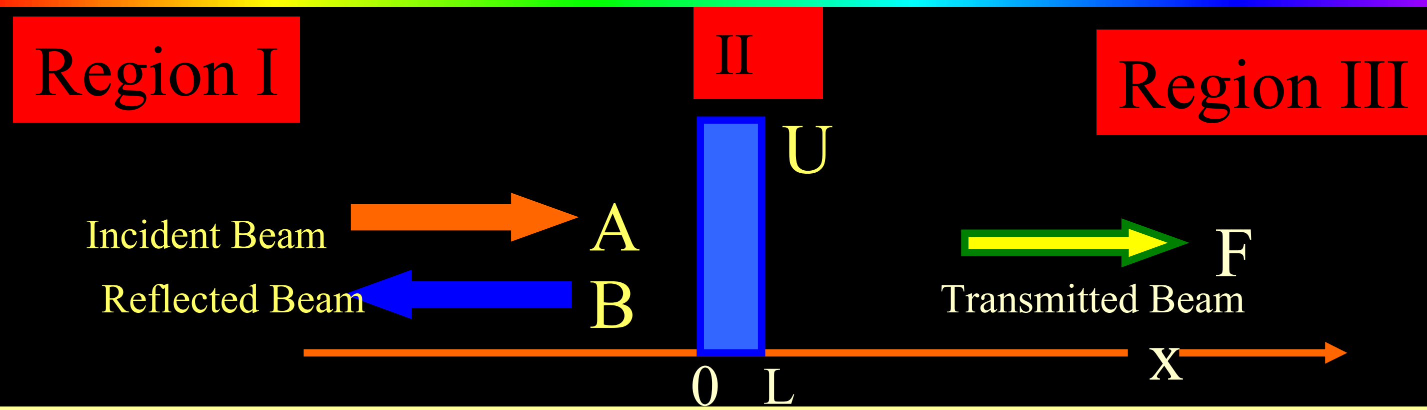
Polonium (Z=84) decays into Lead (Z=82)

$$T(E) = \exp\left\{-4\pi(\boxed{82})\sqrt{(0.0993/8.95)} + 8\sqrt{\boxed{82}(9.00/7.25)}\right\}$$

$$= 8.2\times10^{-13}$$

Taking f=$10^{21}\,Hz \Rightarrow \lambda = 8.2\times10^{-8}$ α emission $\Rightarrow t_{1/2} = \dfrac{0.693}{8.2\times10^{-8}} = 8.4\times10^{-10}\,s$!!!

Potential Barrier : An Unintuitive Result When E>U



Description Of WaveFunctions in Various regions: Simple Ones first

In Region I : $\Psi_{I}(x,t) = Ae^{i(kx-\omega t)} + Be^{i(-kx-\omega t)}$; In Region III: $\Psi_{III}(x,t) = Fe^{i(kx-\omega t)}$

In Region II of Potential U:

$$\text{TISE: } -\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + U\psi(x) = E\psi(x) \Rightarrow \frac{d^2\psi(x)}{dx^2} = \frac{2m}{\hbar^2}(U-E)\psi(x) = \alpha^2\psi(x)$$

with $\alpha^2 = \frac{\sqrt{2m(U-E)}}{\hbar}$; U<E $\Rightarrow \alpha^2 < 0$

Define $\alpha = ik'; \alpha^2 = -(k')^2; k' = \sqrt{\frac{2m(E-U)}{\hbar^2}}$

$\Rightarrow \Psi_{II} = Ce^{i(-k'x-\omega t)} + De^{i(k'x-\omega t)} \Rightarrow$ Oscillatory Wavefunction

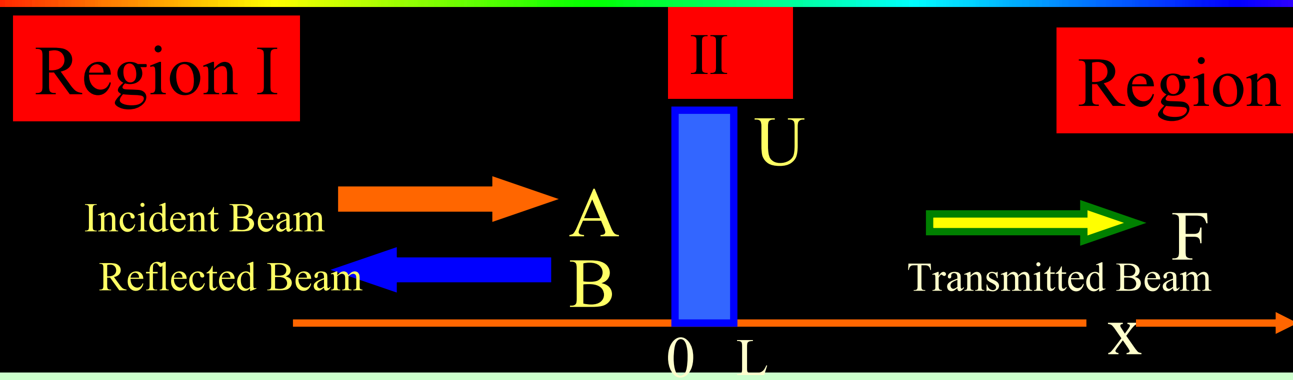
Apply continuity condition at x=0 & x=L

$$\boxed{A+B=C+D};\quad \boxed{kA - kB = k'D - k'C};\quad \boxed{Ce^{-ik'L} + De^{ik'L} = Fe^{ikL}};\quad \boxed{k'De^{ik'L} - k'Ce^{-ik'L} = kFe^{ikL}}$$

Eliminate B, C,D and write every thing in terms of A and F $\Rightarrow$

$$A = \frac{1}{4}Fe^{ikL}\left\{\left[2-\left(\frac{k'}{k}+\frac{k}{k'}\right)\right]e^{ik'L} + \left[2+\left(\frac{k'}{k}+\frac{k}{k'}\right)\right]e^{-ik'L}\right\}$$

Potential Barrier : An Unintuitive Result When E>U



$$\Rightarrow \frac{1}{T} = \frac{A^*A}{F^*F} = \frac{1}{4}\left[2\cos k'L - i\left(\frac{k'}{k} + \frac{k}{k'}\right)\sin k'L\right]^2 = 1 + \frac{1}{4}\left[\frac{U^2}{E(E-U)}\right]\sin^2 k'L > 1$$

Only when $\sin k'L = 0, T = 1$; this happens when $\boxed{k'L = n\pi}$

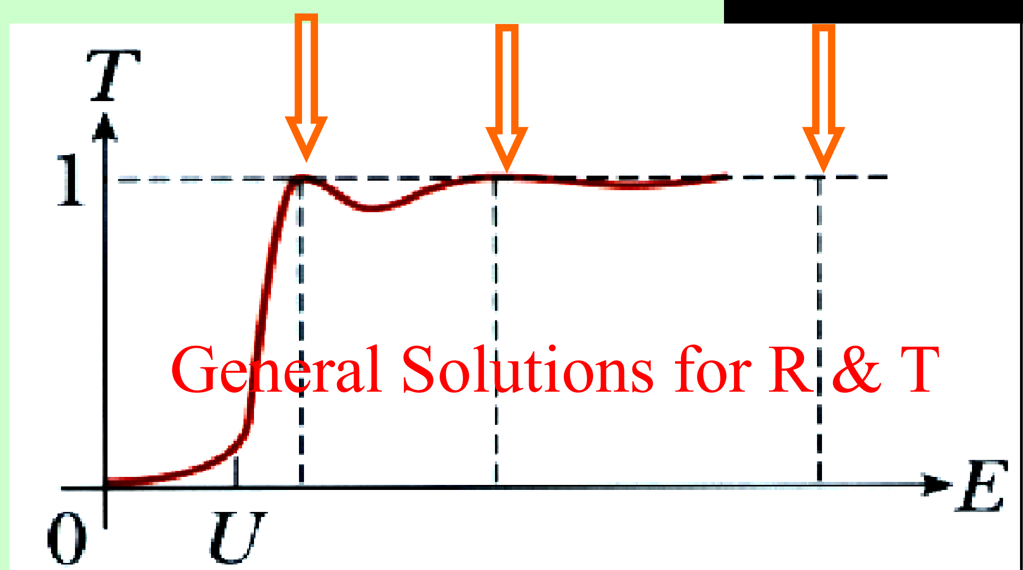
Since $k' = \sqrt{\frac{2m(E-U)}{\hbar^2}} \Rightarrow \sqrt{\frac{2m(E-U)}{\hbar^2}} = n\pi$

$\Rightarrow \boxed{E_n = U + n^2\left(\frac{\pi^2\hbar^2}{2mL^2}\right)}$ is the condition

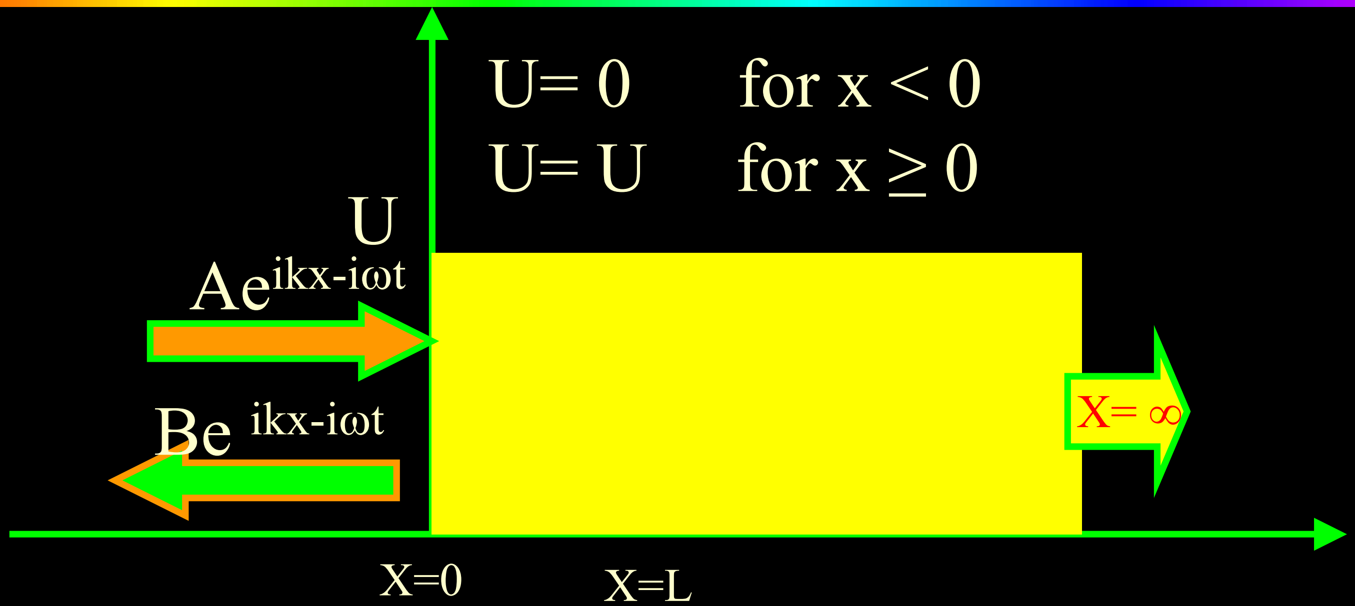
for particle to be completely transmitted

For all other energies, T<1 and R>0 !!!

This is Quantum Mechanics in your face !



Special Case: A Potential Step



In region I (X<0) : $\Psi_{I}(x,t) = Ae^{ikx-i\omega t} + Be^{-ikx-i\omega t}$

In region II (X ≥ 0) : $\Psi_{II}(x,t) = Ce^{-\alpha x-i\omega t}$ $\boxed{+De^{\alpha x-i\omega t}}$

Applying Continuity conditions of Ψ and $\frac{d\Psi}{dx}$ at x=0

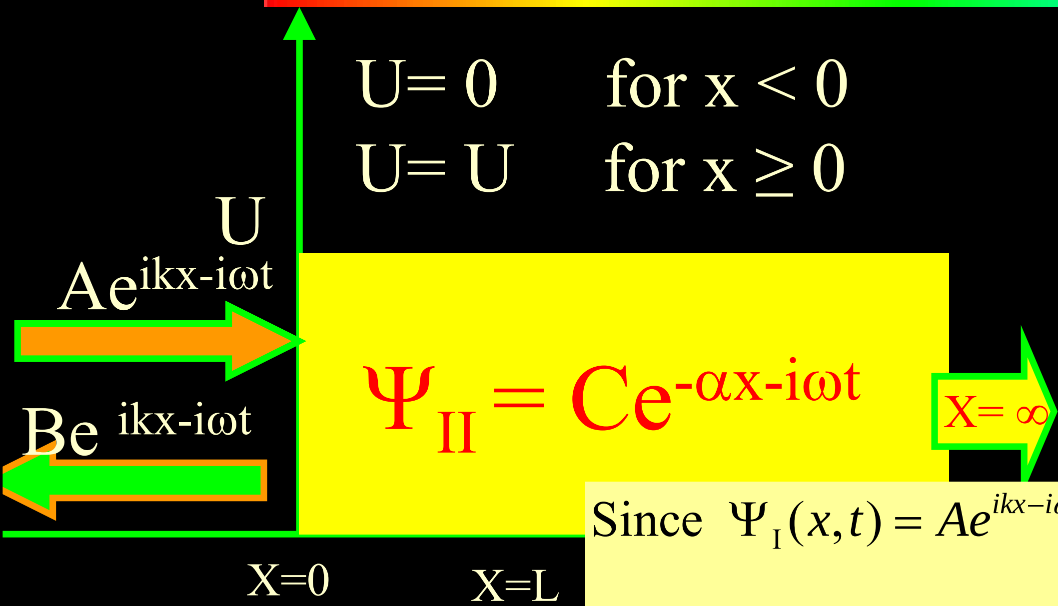
A + B = C & ikA - ikB = -αC; Eliminating C $\Rightarrow$ ikA - ikB =-$\alpha(A+B)$

Defining Penetration Depth $\delta = \frac{1}{\alpha} \Rightarrow \frac{\hbar}{\sqrt{2m(U-E)}}$,

rewrite as ikδA - ikδB = - (A+B) $\Rightarrow$ A(1+ikδ) = -B(1-ikδ)

$\Rightarrow \frac{B}{A} = -\frac{(1+ik\delta)}{(1-ik\delta)}$ $\Rightarrow$ Reflection Coeff $R=\frac{B^{*}B}{A^{*}A}$=1 ; as expected

Transmission Probability in A Potential Step



Since $\Psi_{\rm I}(x,t) = Ae^{ikx-i\omega t} + Be^{-ikx-i\omega t}$; $\Psi_{\rm II}(x,t) = Ce^{-\alpha x-i\omega t}$

Applying Continuity conditions of Ψ and $\frac{d\Psi}{dx}$ at x=0 :

$$A + B = C \Rightarrow \frac{C}{A} = 1 + \frac{B}{A} = 1 - \frac{(1+ik\delta)}{(1-ik\delta)}$$

$$\Rightarrow \frac{C}{A} = -\frac{2ik\delta}{1-ik\delta} \neq 0 \quad \Rightarrow \boxed{T = \left(\frac{C}{A}\right)\left(\frac{C}{A}\right)^{*} > 0!!!}$$

The particle burrows into the skin of the step barrier. If one has a barrier of width L=δ, particle escapes thru the barrier.

penetration distance Δx= distance for which prob. drops by 1/e.

$$|\psi(x=\Delta x)|^2 = C^2 e^{-2\alpha\Delta x} = C^2 e^{-1}$$; happens when $2\alpha\Delta x=1$ or $\boxed{\Delta x = \frac{1}{2}\frac{\hbar}{\sqrt{2m(U-E)}}}$