

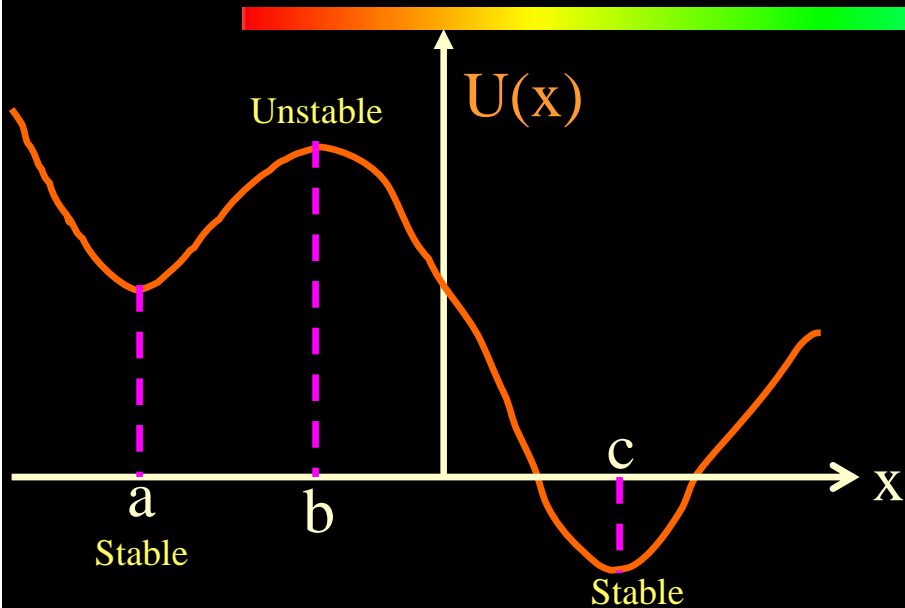
4E : The Quantum Universe



Lecture 16, April 27

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Stable Equilibrium: General Form :

$$U(x) = U(a) + \frac{1}{2}k(x - a)^2$$

$$\text{Rescale} \Rightarrow U(x) = \frac{1}{2}k(x - a)^2$$

Motion of a Classical Oscillator (ideal)

Ball originally displaced from its equilibrium position, motion confined between $x=0$ & $x=A$

$$U(x) = \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2 x^2; \omega = \sqrt{\frac{k}{m}} = \text{Ang. Freq}$$

$$E = \frac{1}{2}kA^2 \Rightarrow \text{Changing } A \text{ changes } E$$

E can take any value & if $A \rightarrow 0$, $E \rightarrow 0$

Max. KE at $x = 0$, KE = 0 at $x = \pm A$

Particle of mass m within a potential $U(x)$

$$\vec{F}(x) = - \frac{dU(x)}{dx}$$

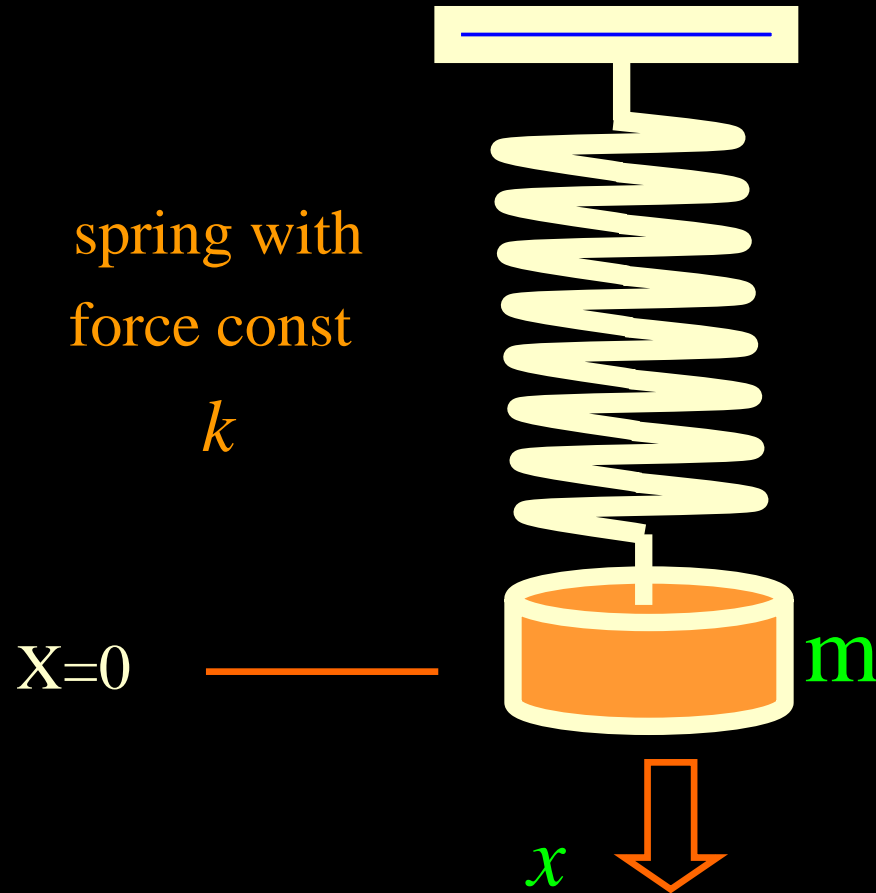
$$\vec{F}(x=a) = - \left. \frac{dU(x)}{dx} \right|_{x=a} = 0,$$

$$\vec{F}(x=b) = 0, \vec{F}(x=c) = 0 \dots \text{But...}$$

look at the Curvature:

$$\frac{\partial^2 U}{\partial x^2} > 0 \text{ (stable)}, \frac{\partial^2 U}{\partial x^2} < 0 \text{ (unstable)}$$

Simple Harmonic Oscillator: Quantum and Classical



Quantum Picture: Harmonic Oscillator

Find the Ground state Wave Function $\psi(x)$

Find the Ground state Energy E when $U(x) = \frac{1}{2}m\omega^2 x^2$

Time Dependent Schrodinger Eqn:
$$\frac{-\hbar^2}{2m} \frac{\partial^2 \psi(x)}{\partial x^2} + \frac{1}{2}m\omega^2 x^2 \psi(x) = E \psi(x)$$

$$\Rightarrow \frac{d^2 \psi(x)}{dx^2} = \frac{2m}{\hbar^2} \left(E - \frac{1}{2}m\omega^2 x^2 \right) \psi(x) = 0$$
 What $\psi(x)$ solves this?

Two guesses about the simplest Wavefunction:

1. $\psi(x)$ should be symmetric about x 2. $\psi(x) \rightarrow 0$ as $x \rightarrow \infty$

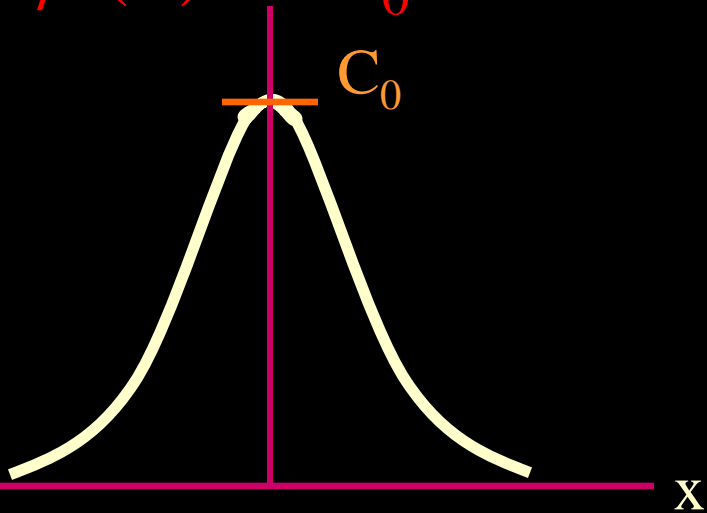
+ $\psi(x)$ should be continuous & $\frac{d\psi(x)}{dx}$ = continuous

My guess: $\psi(x) = C_0 e^{-\alpha x^2}$; Need to find C_0 & α :

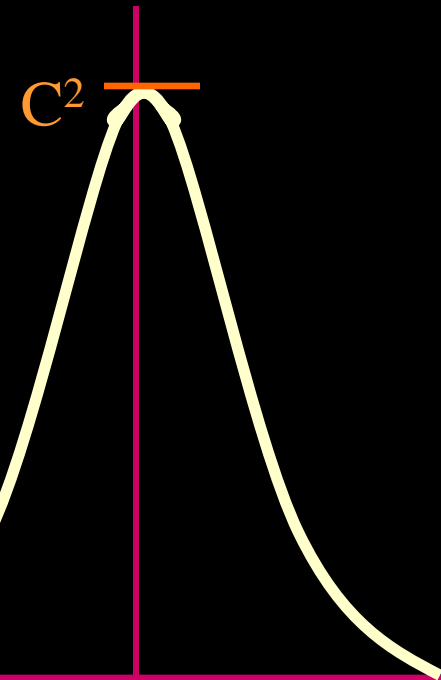
What does this wavefunction & PDF look like?

Quantum Picture: Harmonic Oscillator

$$\psi(x) = C_0 e^{-\alpha x^2}$$



$$P(x) = C_0^2 e^{-2\alpha x^2}$$



How to Get C_0 & α ?? ...Try plugging in the wave-function into the time-independent Schr. Eqn.

Time Independent Sch. Eqn & The Harmonic Oscillator

Master Equation is :
$$\frac{\partial^2 \psi(x)}{\partial x^2} = \frac{2m}{\hbar^2} \left[\frac{1}{2} m \omega^2 x^2 - E \right] \psi(x)$$

Since $\psi(x) = C_0 e^{-\alpha x^2}$,
$$\frac{d\psi(x)}{dx} = C_0 (-2\alpha x) e^{-\alpha x^2},$$

$$\frac{d^2 \psi(x)}{dx^2} = C_0 \frac{d(-2\alpha x)}{dx} e^{-\alpha x^2} + C_0 (-2\alpha x)^2 e^{-\alpha x^2} = C_0 [4\alpha^2 x^2 - 2\alpha] e^{-\alpha x^2}$$

$$\Rightarrow C_0 [4\alpha^2 x^2 - 2\alpha] e^{-\alpha x^2} = \frac{2m}{\hbar^2} \left[\frac{1}{2} m \omega^2 x^2 - E \right] C_0 e^{-\alpha x^2}$$

Match the coeff of x^2 and the Constant terms on LHS & RHS

$$\Rightarrow 4\alpha^2 = \frac{2m}{\hbar^2} \frac{1}{2} m \omega^2 \quad \text{or} \quad \alpha = \frac{m\omega}{2\hbar}$$

& the other match gives $2\alpha = \frac{2m}{\hbar^2} E$, substituing $\alpha \Rightarrow$

$$E = \frac{1}{2} \hbar \omega = hf \quad \text{!!!!.....(Planck's Oscillators)}$$

What about C_0 ? We learn about that from the Normalization cond.

SHO: Normalization Condition

$$\int_{-\infty}^{+\infty} |\psi_0(x)|^2 dx = 1 = \int_{-\infty}^{+\infty} C_0^2 e^{-\frac{m\omega x^2}{\hbar}} dx$$

Since $\int_{-\infty}^{+\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}$ (dont memorize this)

Identifying $a = \frac{m\omega}{\hbar}$ and using the identity above

$$\Rightarrow C_0 = \left[\frac{m\omega}{\pi\hbar} \right]^{\frac{1}{4}}$$

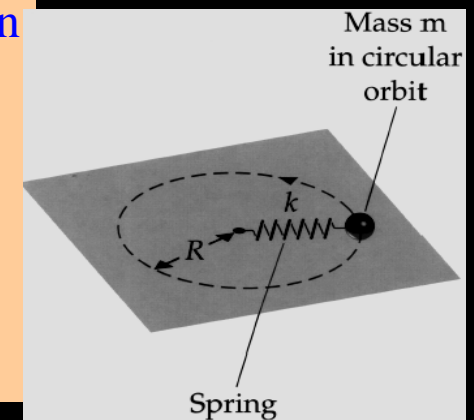
Hence the Complete NORMALIZED wave function is :

$$\psi_0(x) = \left[\frac{m\omega}{\pi\hbar} \right]^{\frac{1}{4}} e^{-\frac{m\omega x^2}{2\hbar}}$$

Ground State Wavefunction

has energy $E = hf$

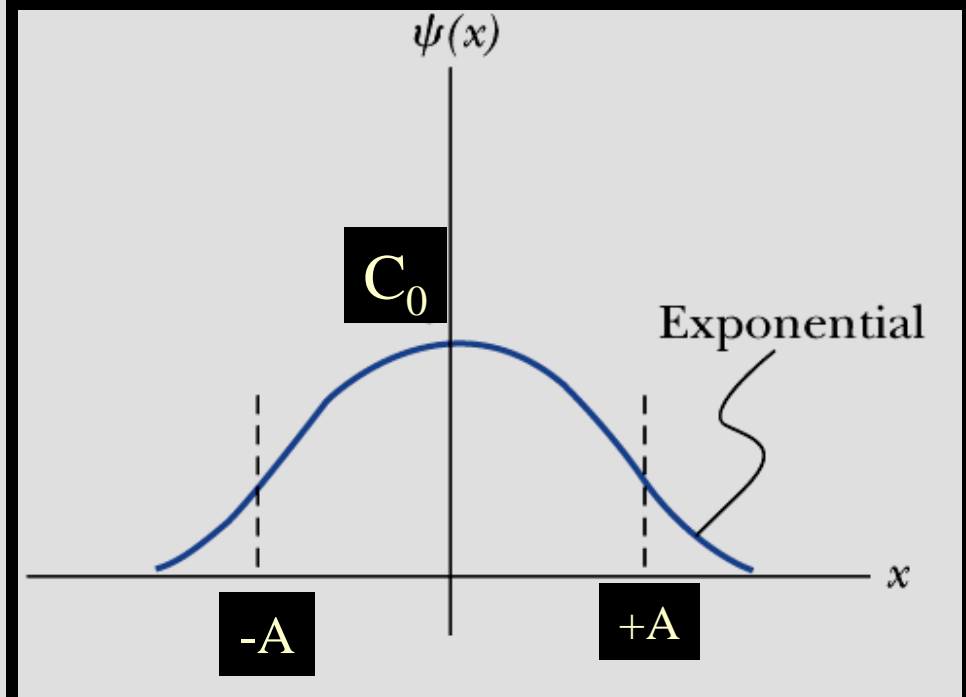
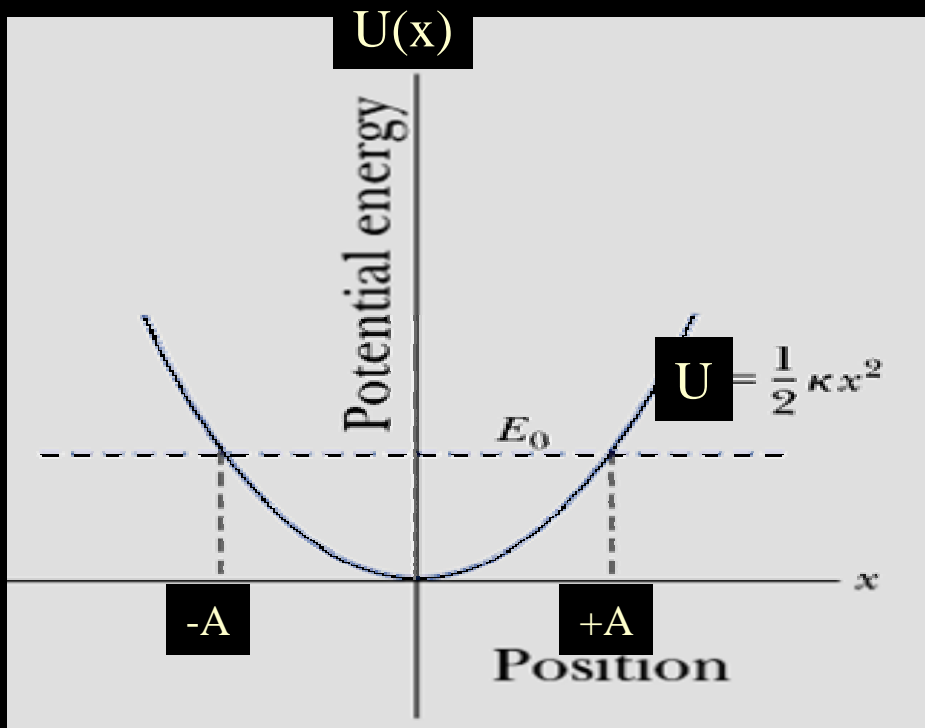
Planck's Oscillators were electrons tied by the "spring" of the mutually attractive Coulomb Force



Quantum Oscillator In Pictures

$$E = KE + U(x) > 0 \text{ for } n=0$$

Quantum Mechanical prob for particle
To live outside classical turning points
Is finite !



Classically particle most likely to be at the turning point (velocity=0)
Quantum Mechanically , particle most likely to be at $x=x_0$ for $n=0$

Classical & Quantum Pictures of SHO compared



- Limits of classical vibration : Turning Points (do on Board)
- Quantum Probability for particle outside classical turning points $P(|x| > A) = 16\%$!!
 - Do it on the board (see Example problems in book)

Excited States of The Quantum Oscillator

$$\psi_n(x) = C_n H_n(x) e^{-\frac{m\omega x^2}{2\hbar}} ;$$

$H_n(x)$ = Hermite Polynomials

with

$$H_0(x) = 1$$

$$H_1(x) = 2x$$

$$H_2(x) = 4x^2 - 2$$

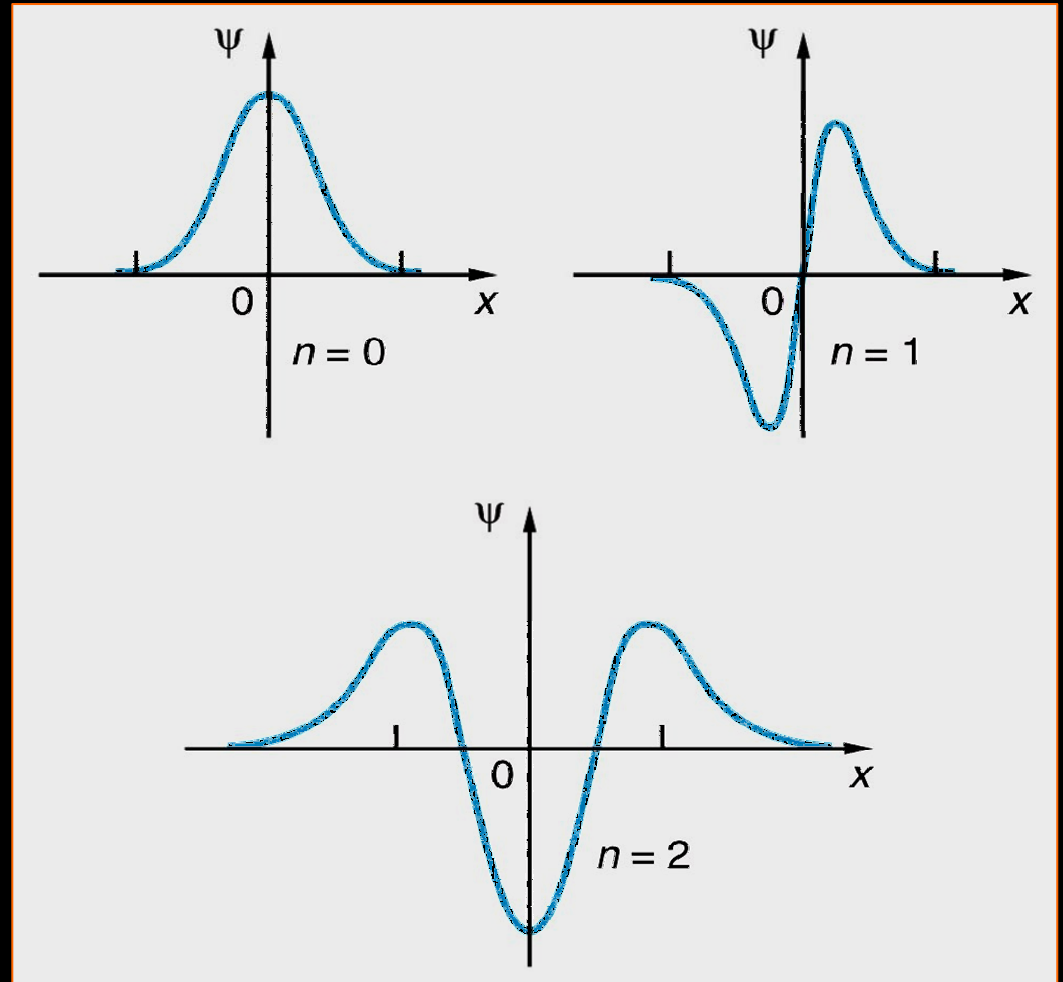
$$H_3(x) = 8x^3 - 12x$$

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n e^{-x^2}}{dx^n}$$

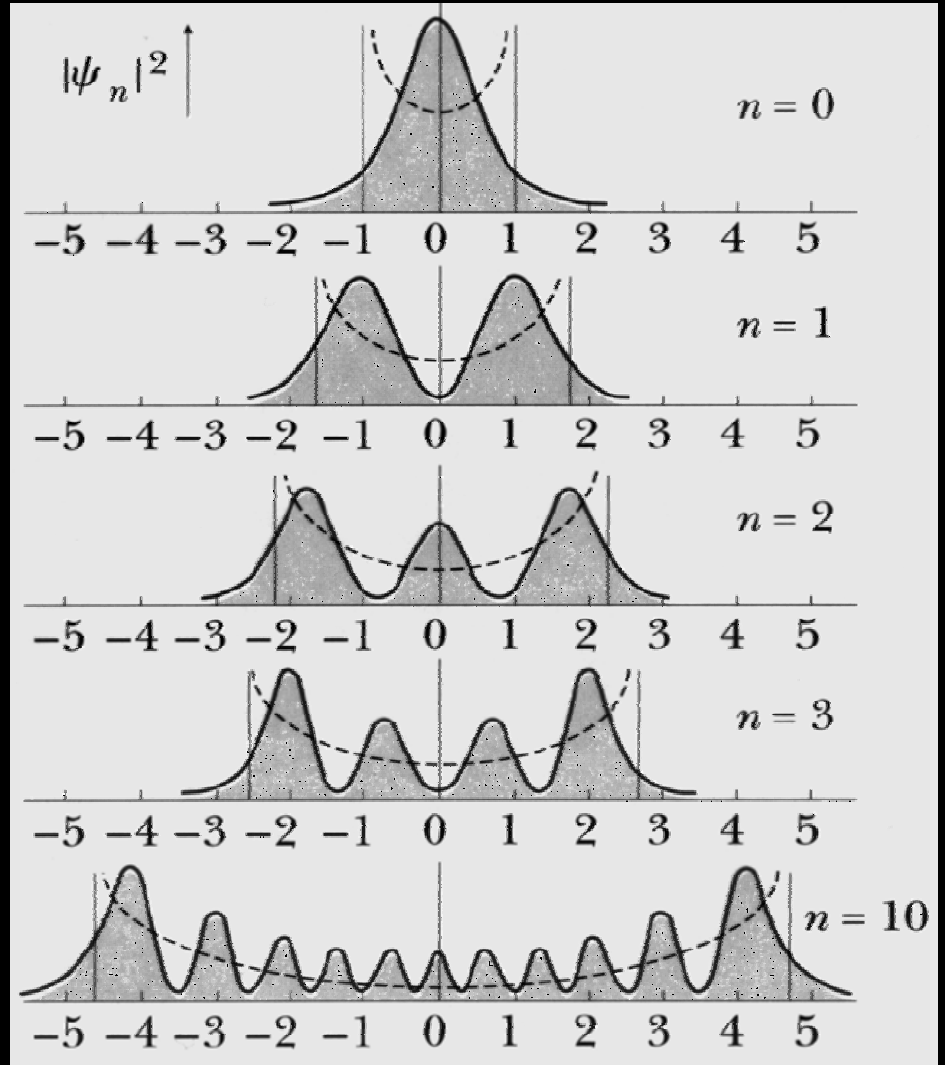
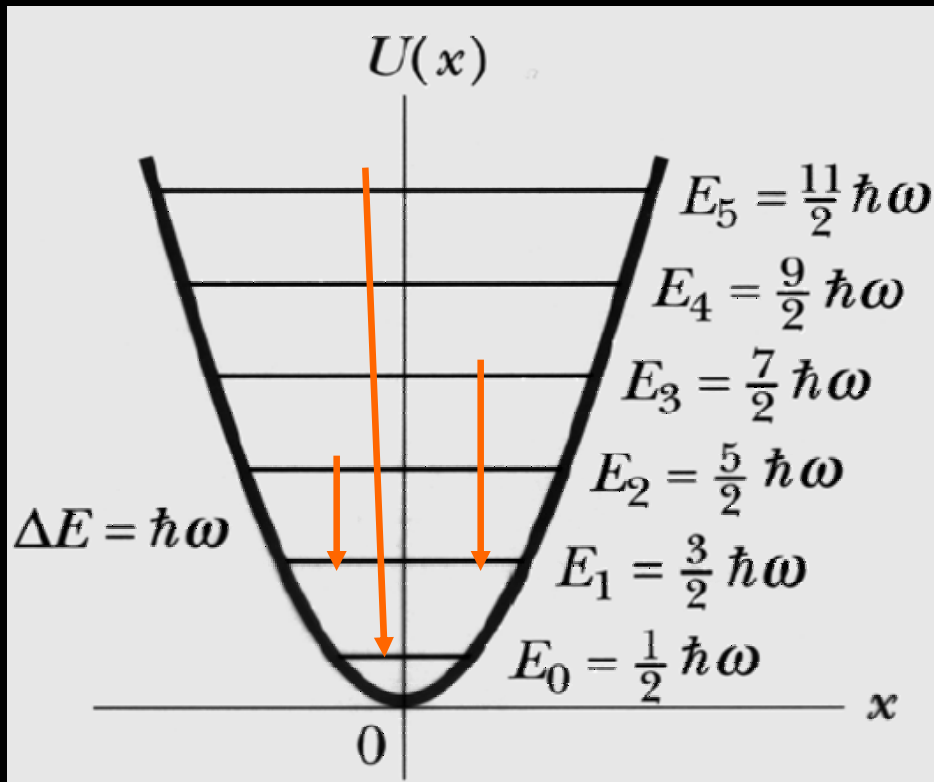
and

$$E_n = \left(n + \frac{1}{2}\right) \hbar \omega = \left(n + \frac{1}{2}\right) hf$$

Again $n=0,1,2,3,\dots,\infty$ Quantum #



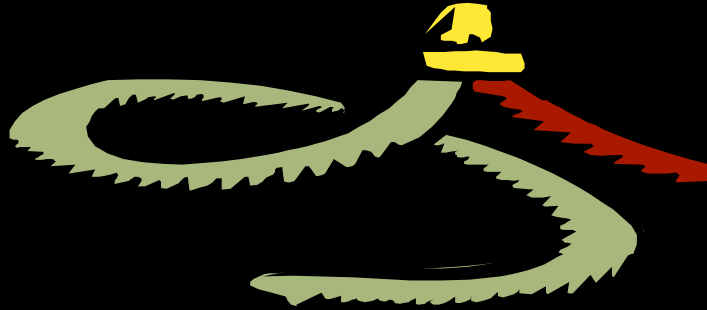
Excited States of The Quantum Oscillator



Ground State Energy > 0 always

As $n \rightarrow \infty$ classical and quantum probabilities become similar

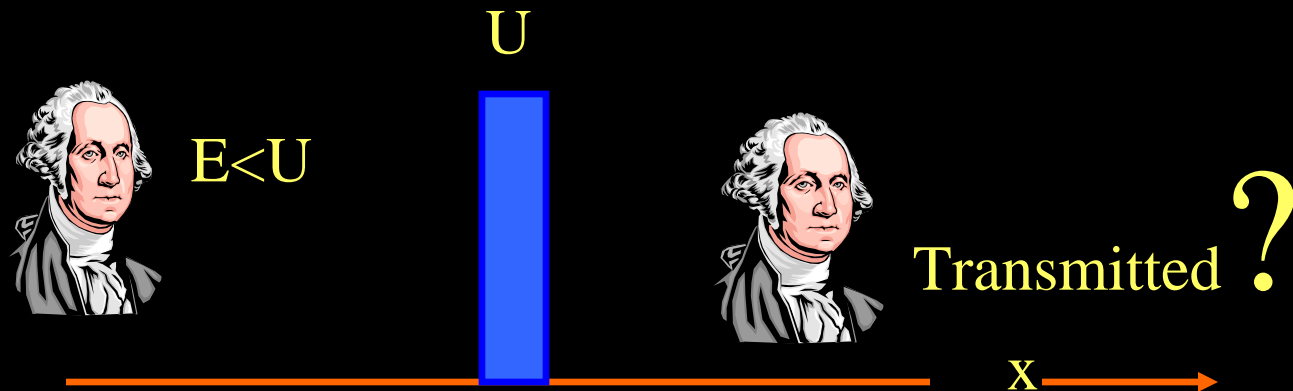
The Case of a Rusty “Twisted Pair” of Naked Wires & How Quantum Mechanics Saved ECE Majors !



- Twisted pair of Cu Wire (metal) in virgin form
- Does not stay that way for long in the atmosphere
 - Gets oxidized in dry air quickly $\text{Cu} \rightarrow \text{Cu}_2\text{O}$
 - In wet air $\text{Cu} \rightarrow \text{Cu}(\text{OH})_2$ (the green stuff on wires)
- Oxides or Hydride are non-conducting ..so no current can flow across the junction between two metal wires
- No current means no circuits \rightarrow no EE, no ECE !!
- All ECE majors must now switch to Chemistry instead
& play with benzene !!! Bad news !



Potential Barrier



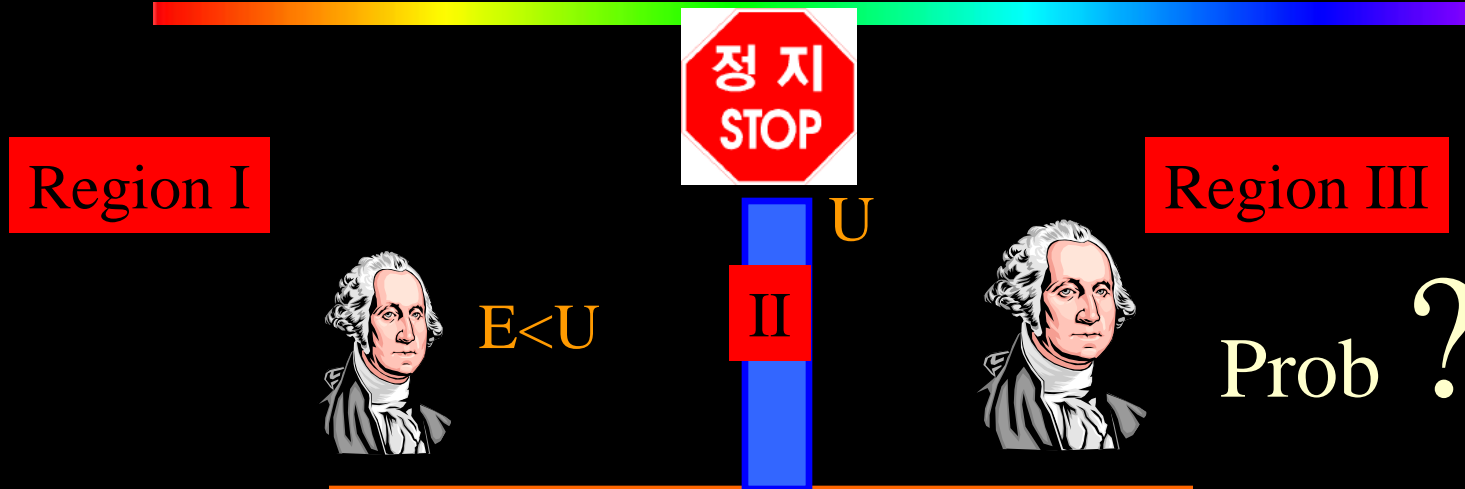
Description of Potential

$U = 0$	$x < 0$	(Region I)
$U = U$	$0 < x < L$	(Region II)
$U = 0$	$x > L$	(Region III)

Consider George as a “free Particle/Wave” with Energy E incident from Left
Free particle are under no Force; have wavefunctions like

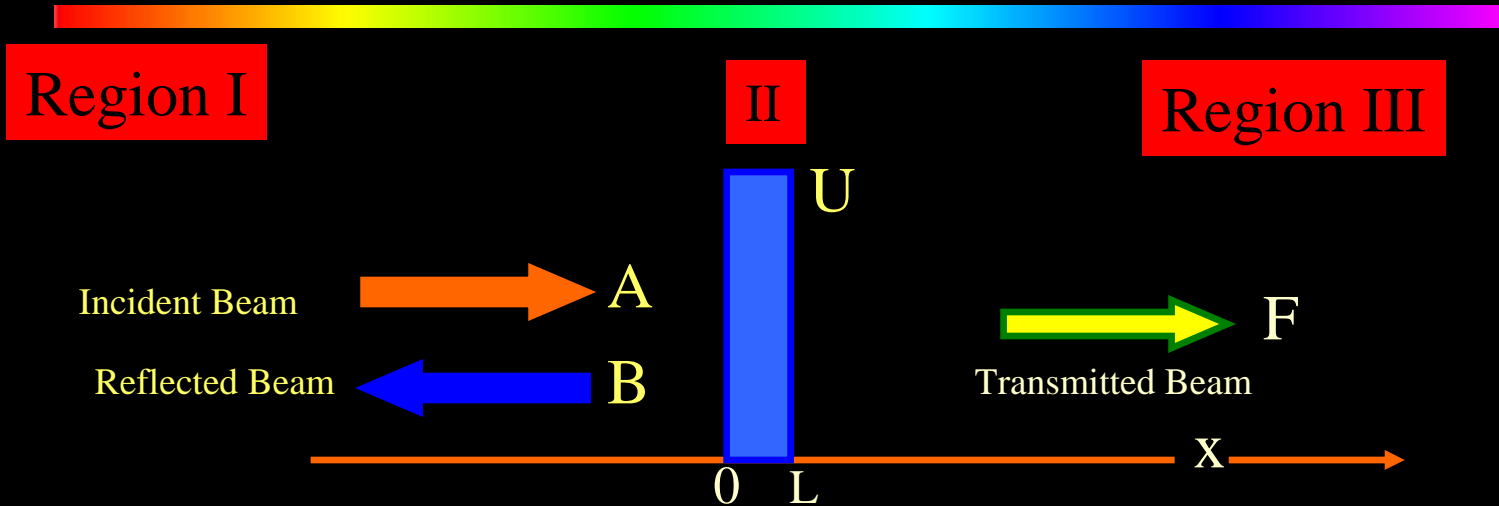
$$\Psi = A e^{i(kx - \omega t)} \text{ or } B e^{i(-kx - \omega t)}$$

Tunneling Through A Potential Barrier



- Classical & Quantum Pictures compared: When $E > U$ & when $E < U$
- Classically, an particle or a beam of particles incident from left encounters barrier:
 - when $E > U \rightarrow$ Particle just goes over the barrier (gets transmitted)
 - When $E < U \rightarrow$ particle is stuck in region I, gets entirely reflected, no transmission (T)
- What happens in a Quantum Mechanical barrier? No region is inaccessible for particle since the potential is (sometimes small) but finite

Beam Of Particles With $E < U$ Incident On Barrier From Left



Description Of WaveFunctions in Various regions: Simple Ones first

In Region I: $\Psi_I(x,t) = Ae^{i(kx-\omega t)} + Be^{i(-kx-\omega t)}$ = incident + reflected Waves

$$\text{with } E = \hbar\omega = \frac{\hbar^2 k^2}{2m}$$

define Reflection Coefficient: $R = \frac{|B|^2}{|A|^2}$ = frac of incident wave intensity reflected back

In Region III: $\Psi_{III}(x,t) = Fe^{i(kx-\omega t)} + Ge^{i(-kx-\omega t)}$ = transmitted

Note: $Ge^{i(-kx-\omega t)}$ corresponds to wave incident from right !

This piece does not exist in the scattering picture we are thinking of now ($G=0$)

So $\Psi_{III}(x,t) = Fe^{i(kx-\omega t)}$ represents transmitted beam. Define $T = \frac{|F|^2}{|A|^2}$

Unitarity Condition $\Rightarrow R + T = 1$ (particle is either reflected or transmitted)

Wave Function Across The Potential Barrier

In Region II of Potential U

$$\text{TISE: } -\frac{\hbar^2}{2m} \frac{d^2\psi(x)}{dx^2} + U\psi(x) = E\psi(x)$$

$$\Rightarrow \frac{d^2\psi(x)}{dx^2} = \frac{2m}{\hbar^2} (U - E)\psi(x) = \alpha^2\psi(x)$$

$$\text{with } \alpha^2 = \frac{\sqrt{2m(U-E)}}{\hbar} ; U > E \Rightarrow \alpha^2 > 0$$

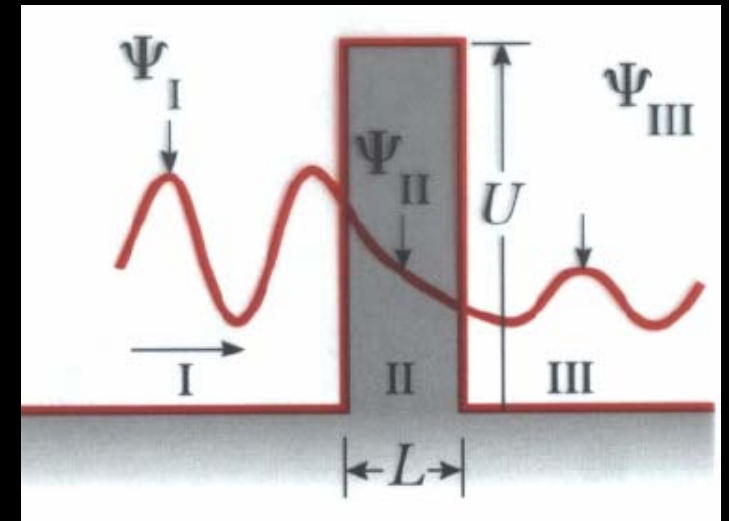
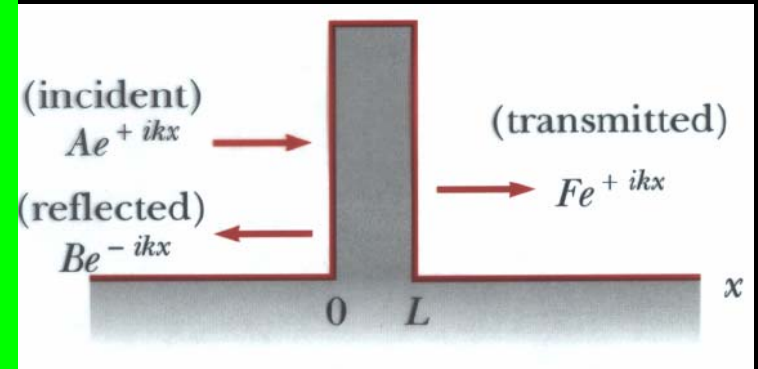
Solutions are of form $\psi(x) \propto e^{\pm\alpha x}$

$$\Psi_{II}(x,t) = Ce^{+\alpha x - i\omega t} + De^{-\alpha x - i\omega t} \quad 0 < x < L$$

To determine C & D \Rightarrow apply matching cond.

$\Psi_{II}(x,t) = \text{continuous}$ across barrier (x=0,L)

$$\frac{d\Psi_{II}(x,t)}{dx} = \text{continuous across barrier (x=0,L)}$$



Continuity Conditions Across Barrier

At $x = 0$, continuity of $\psi(x) \Rightarrow$

$$A+B=C+D \quad (1)$$

At $x = 0$, continuity of $\frac{d\psi(x)}{dx} \Rightarrow$

$$ikA - ikB = \alpha C - \alpha D \quad (2)$$

Similarly at $x=L$ continuity of $\psi(x) \Rightarrow$

$$Ce^{-\alpha L} + De^{+\alpha L} = Fe^{ikL} \quad (3)$$

at $x=L$, continuity of $\frac{d\psi(x)}{dx} \Rightarrow$

$$-(\alpha C)e^{-\alpha L} + (\alpha D)e^{+\alpha L} = ikFe^{ikL} \quad (4)$$

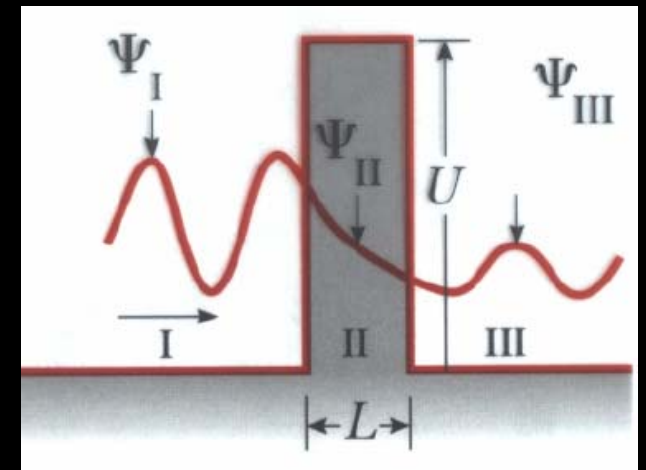
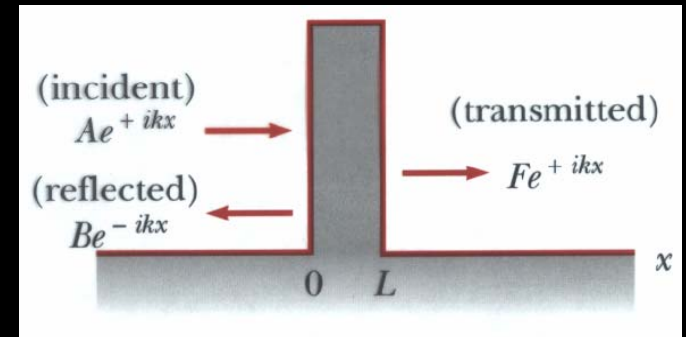
Four equations & four unknowns

Cant determine A,B,C,D but if you

Divide thruout by A in all 4 equations :

\Rightarrow ratio of amplitudes \rightarrow relations for R & T

That's what we need any way



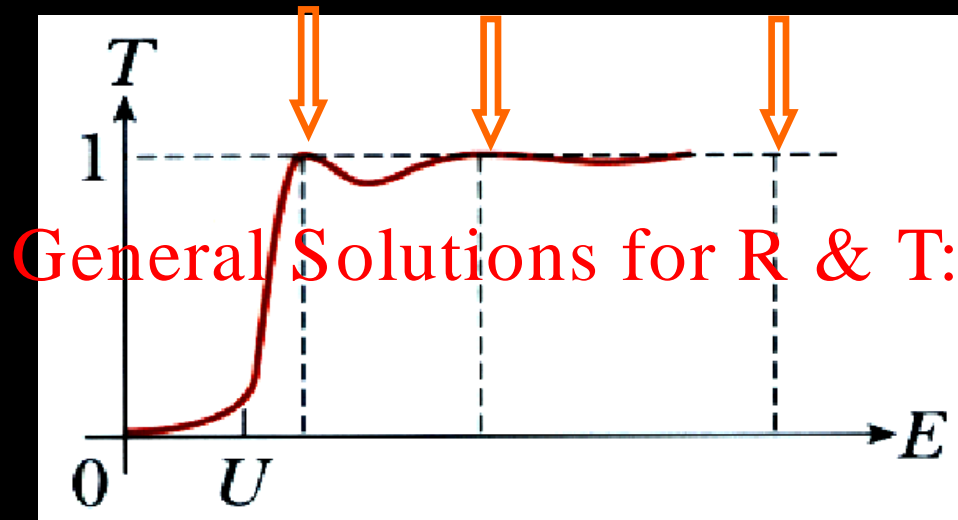
Potential Barrier when $E < U$

Expression for Transmission Coeff $T=T(E)$:

Depends on barrier Height U , barrier Width L and particle Energy E

$$T(E) = \left[1 + \frac{1}{4} \left(\frac{U^2}{E(U-E)} \right) \sinh^2(\alpha L) \right]^{-1}; \quad \alpha = \frac{\sqrt{2m(U-E)}}{\hbar}$$

and $R(E)=1-T(E)$what's not transmitted is reflected



Above equation holds only for $E < U$

For $E > U$, $\alpha = \text{imaginary}$

$\text{Sinh}(\alpha L)$ becomes oscillatory

This leads to an Oscillatory $T(E)$ and

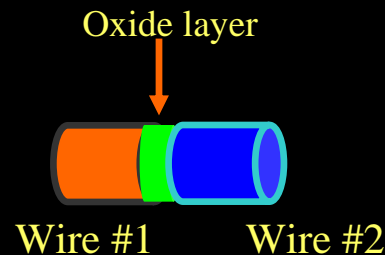
Transmission resonances occur where

For some specific energy ONLY, $T(E) = 1$

At other values of E , some particles are reflected back ..even though $E > U$!!

That's the Wave nature of the Quantum particle

Separated in Coppertino



Q: 2 Cu wires are separated by insulating Oxide layer. Modeling the Oxide layer as a square barrier of height $U=10.0\text{eV}$, estimate the transmission coeff for an incident beam of electrons of $E=7.0\text{ eV}$ when the layer thickness is (a) 5.0 nm (b) 1.0nm

Q: If a 1.0 mA current in one of the intertwined wires is incident on Oxide layer, how much of this current passes thru the Oxide layer on to the adjacent wire if the layer thickness is 1.0nm? What becomes of the remaining current?

$$T(E) = \left[1 + \frac{1}{4} \left(\frac{U^2}{E(U-E)} \right) \sinh^2(\alpha L) \right]^{-1}$$

$$\alpha = \frac{\sqrt{2m(U-E)}}{\hbar}, k = \frac{\sqrt{2mE}}{\hbar}$$

$$T(E) = \left[1 + \frac{1}{4} \left(\frac{U^2}{E(U-E)} \right) \sinh^2(\alpha L) \right]^{-1}$$

Use $\hbar = 1.973 \text{ keV}\cdot\text{\AA}/c$, $m_e = 511 \text{ keV}/c^2$

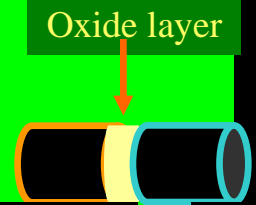
$$\Rightarrow \alpha = \frac{\sqrt{2m_e(U-E)}}{\hbar} = \frac{\sqrt{2 \times 511 \text{ keV}/c^2 (3.0 \times 10^{-3} \text{ keV})}}{1.973 \text{ keV}\cdot\text{\AA}/c} = 0.8875 \text{\AA}^{-1}$$

Substitute in expression for $T=T(E)$

$$T = \left[1 + \frac{1}{4} \left(\frac{10^2}{7(10-7)} \right) \sinh^2(0.8875 \text{\AA}^{-1})(50 \text{\AA}) \right]^{-1} = 0.963 \times 10^{-38} \text{ (small)!!}$$

However, for $L=10 \text{\AA}$; $T=0.657 \times 10^{-7}$

Reducing barrier width by $\times 5$ leads to Trans. Coeff enhancement by 31 orders of magnitude !!!



Wire #1

Wire #2

$$1 \text{ mA current} = I = \frac{Q = Nq_e}{t} \Rightarrow N = 6.25 \times 10^{15} \text{ electrons}$$

N_T = # of electrons that escape to the adjacent wire (past oxide layer)

$$N_T = N \cdot T = (6.25 \times 10^{15} \text{ electrons}) \times \boxed{T};$$

$$\text{For } L=10 \text{\AA}, T=0.657 \times 10^{-7} \Rightarrow N_T = 4.11 \times 10 \Rightarrow \boxed{I_T = 65.7 \text{ pA}} \text{!!}$$

Current Measured on the first wire is sum of incident+reflected currents and current measured on "adjacent" wire is the I_T

Oxide thickness makes all the difference !
That's why from time-to-time one needs to Scrape off the green stuff off the naked wires

A Special Case That is Instructive & Useful: $U \gg E$

Given the 4 equations from Continuity Conditions: Solve for $\frac{A}{F}$

$$\frac{A}{F} = \left[\frac{1}{2} + \frac{i}{4} \left(\frac{\alpha}{k} - \frac{k}{\alpha} \right) \right] e^{(ik+\alpha)L} + \left[\frac{1}{2} - \frac{i}{4} \left(\frac{\alpha}{k} - \frac{k}{\alpha} \right) \right] e^{(ik-\alpha)L}$$

Remember $\alpha = \frac{\sqrt{2m(U-E)}}{\hbar}, k = \frac{\sqrt{2mE}}{\hbar},$

when $U \gg E$, $\alpha \gg k$ & $\frac{\alpha}{k} \gg \frac{k}{\alpha}$ So $\frac{\alpha}{k} - \frac{k}{\alpha} \approx \frac{\alpha}{k}$; For large Barrier L, $\alpha L \gg 1$

$$\frac{A}{F} = \left[\frac{1}{2} + \frac{i}{4} \left(\frac{\alpha}{k} \right) \right] e^{(ik+\alpha)L}; \left(\frac{A}{F} \right)^* = \left[\frac{1}{2} - \frac{i}{4} \left(\frac{\alpha}{k} \right) \right] e^{(-ik+\alpha)L}$$

$$T^{-1} = \left(\frac{A}{F} \right)^* \frac{A}{F} = \left[\frac{1}{4} + \frac{1}{16} \left(\frac{\alpha^2}{k^2} \right) \right] e^{(2\alpha)L} = \frac{1}{T(E)}; \text{ now invert \& consolidate}$$

$$T = \left| \frac{F^* F}{A^* A} \right| = \left(\frac{16}{4 + \left(\frac{\alpha}{k} \right)^2} \right) e^{-2\alpha L}; \text{ now watch the variables employed}$$

A Special Case That is Instructive & Useful: $U \gg E$

$$\alpha = \frac{\sqrt{2m(U-E)}}{\hbar}, k = \frac{\sqrt{2mE}}{\hbar} = \frac{p}{\hbar} = \frac{2\pi}{\lambda}$$

$$\left(\frac{\alpha}{k}\right)^2 = \frac{2m(U-E)/\hbar^2}{2mE/\hbar^2} = \frac{U}{E} - 1 \approx \frac{U}{E}$$

$$\Rightarrow \left(\frac{16}{4 + \left(\frac{\alpha}{k}\right)^2}\right) = \left(\frac{16}{4 + \left(\frac{U}{E} - 1\right)}\right) \rightarrow \text{varies slowly compared with } e^{-2\alpha L} \text{ term}$$

keeping in mind only the Order of magnitude, I suggest

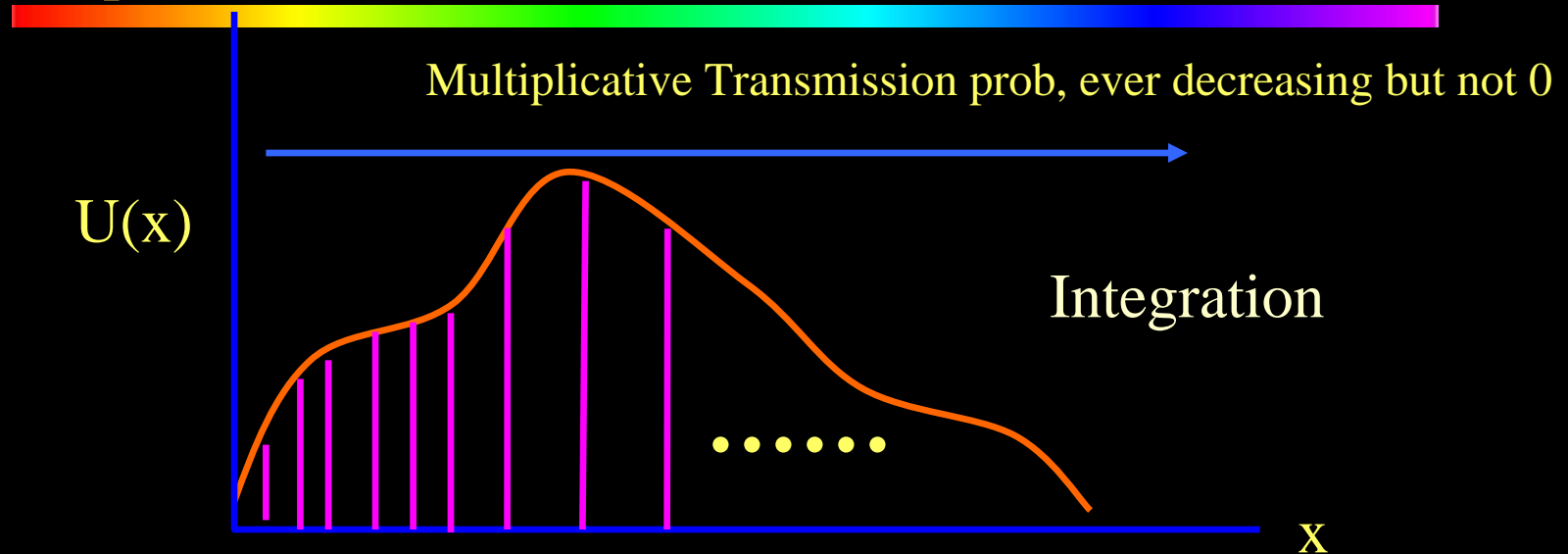
$$\left(\frac{16}{4 + \left(\frac{U}{E} - 1\right)}\right) \approx 1; \text{ back to } T = \left(\frac{16}{4 + \left(\frac{\alpha}{k}\right)^2}\right) e^{-2\alpha L} \text{ substituting}$$

So approximately $T \approx e^{-2\alpha L}$ Transmission Prob is fn of U,E,L

Why subject you to this TORTURE? → Estimate T for complicated Potentials

See next (example of Cu oxide layer, radioactivity, blackhole blowup etc)

A Complicated Potential Barrier Can Be Broken Down



Can be broken down into a sum of successive Rectangular potential barriers for which we learnt to find the Transmission probability T_i

The Transmitted beam intensity thru one small barrier becomes incident beam intensity for the following one

So on & so forth ...till the particle escapes with final Transmission prob T

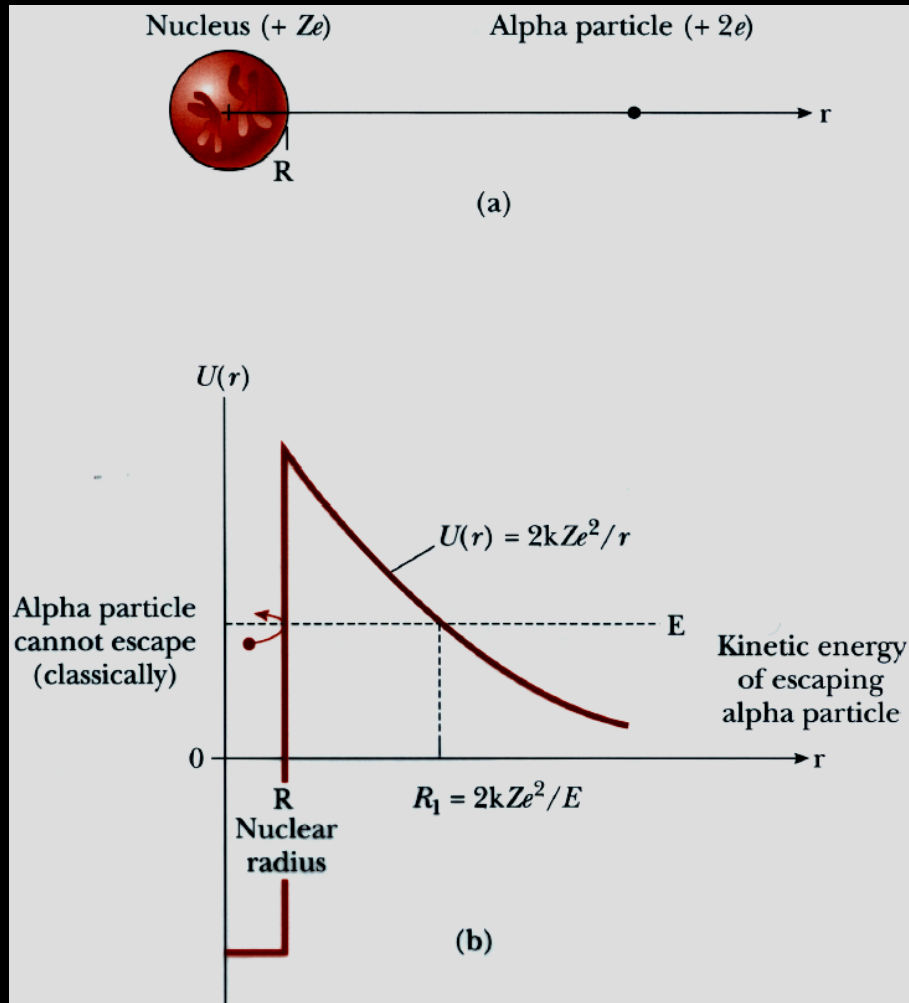
$$T = \int T_i dx = e^{-2 \left[\frac{\sqrt{2m}}{\hbar} \int \sqrt{U(x) - E} dx \right]}$$

The Great Escape ! My Favorite Movie



Story involves an Allied plan for a massive breakout from a Nazi P.O.W. camp, during World War Two. The Nazis had created a high-security, escape -proof prisoner of war camp for those annoying detainees who have attempted escape from their other prison of war camps. These prisoners are not discouraged at all, as they plan a huge escape of 100 men.

Radioactivity: The α -particle & Steve McQueen Compared



- In a Nucleus such as Ra, Uranium etc α particle rattles around parent nucleus, “hitting” the nuclear walls with a very high frequency (probing the “fence”), if the transmission prob $T > 0$, then eventually particle escapes
- Within nucleus, α particle is virtually free but is trapped by the Strong nuclear force (see quiz), once outside nucleus, the particle “sees” only the columbic force (nuclear force too faint outside)
- Nuclear radius $R = 10^{-14}$ m, $E_\alpha = 9$ MeV
- Coulomb barrier $U(r) = kq_1q_2/r$
 - At $r=R$, $U(R) \approx 30$ MeV barrier
- α -particle, due to QM, tunnels thru
- It’s the sensitivity of T on E_α that accounts for the wide range in half-lives of radioactive nuclei

$$T = \int T_i dx = e^{-2 \left[\frac{\sqrt{2m_\alpha}}{\hbar} \int \sqrt{\frac{2ke^2Z}{r} - E_\alpha} dx \right]}$$