Department of Physics
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Quantum Universe (4E)
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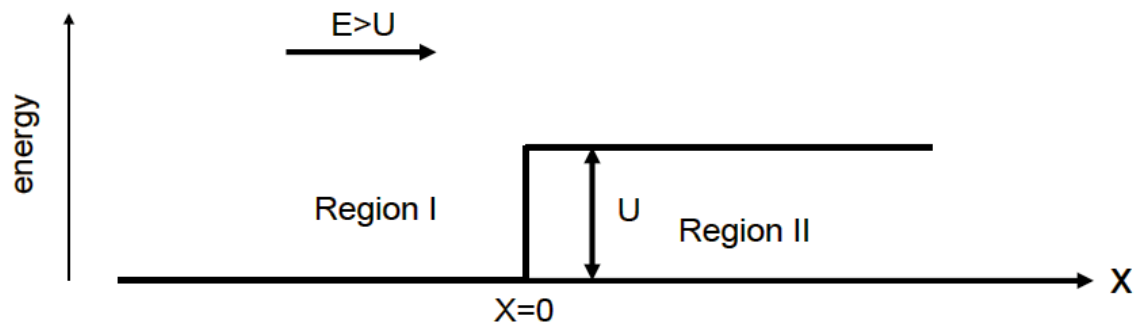
Quiz # 3(May 7 2004)

Problem 1 : A Step in The Right Direction [9 pts]

Consider a beam of particles of kinetic energy E incident from the left on a potential step of height U starting at x=0. The particles have energy E > U. (a) write the physical solutions of the time independent S. equation for each region. (b) By applying the boundary condition, show that the expression for transmission probability is $T = \frac{4k_I k_{II}}{(k_I + k_{II})^2}$ (c) A 1.0 mA beam of electrons moving at $2.0\times10^6 m/s$ is incident from left, at the boundary (x=0) the speed of the beam is reduced to $1.0\times10^6 m/s$ due to the difference in potential. Calculate the transmitted and reflected currents. Use the notation $k_I^2 = \frac{2mE}{\hbar^2}, k_{II}^2 = \frac{2m(E-U)}{\hbar^2}$.



Problem 2: Lazy "R" Us ! [11 pts]

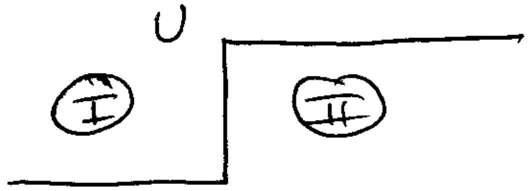
The wavefunction for the ground state of a 1D harmonic oscillator is given by $\psi(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega x^2}{2\hbar}}$. The energy of this state is E=$\frac{1}{2}\hbar\omega$. Calculate (a) <x>, (b) <x^2> and (c) Δx for this state. (d) calculate <p>. (e) Use energy conservation to relate <p^2> to <x^2>, and calculate <p^2>. (f) calculate Δp and, finally, (g) calculate the product Δx.Δp .

Hint: If possible, avoid integration! If you must, use

$$\int_0^\infty x^2 e^{-ax^2} dx = \frac{1}{4a}\sqrt{\frac{\pi}{a}} \quad \text{[for a>0]}$$

Phys 4E Quiz 3

1]



a] Sch. Eq: $-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V\psi = E\psi$

I: $\frac{\partial^2\psi}{\partial x^2} = -\frac{2mE}{\hbar^2}\psi \Rightarrow \boxed{\psi_I = Ae^{ik_I x} + Be^{-ik_I x}}$ $k_I^2 = \frac{2mE}{\hbar^2}$

II $\frac{\partial^2\psi}{\partial x^2} = -\frac{2m(E-U)}{\hbar^2}\psi \Rightarrow \psi_{II} = Ce^{ik_{II}x} + De^{-ik_{II}x}$, $k_{II}^2 = \frac{2m(E-U)}{\hbar^2}$

Lose D b/c nothing to reflect off $\Rightarrow \boxed{\psi_{II} = Ce^{ik_{II}x}}$

b] Continuous: $A + B = C$

Smooth: $k_I(A - B) = k_{II}C$

Solve for C, B: $A + B = C$, $A - B = \frac{k_{II}}{k_I}C$ $\Rightarrow 2A = \left(1 + \frac{k_{II}}{k_I}\right)C$

So $C = \frac{2k_I}{k_I + k_{II}}A$, $B = C - A = \frac{k_I - k_{II}}{k_I + k_{II}}$

$R = \left|\frac{B}{A}\right|^2 = \left(\frac{k_I - k_{II}}{k_I + k_{II}}\right)^2$, $\boxed{T = 1 - R = \frac{4k_I k_{II}}{(k_I + k_{II})^2}}$

c] Since $p = \hbar k$, $\dfrac{k_I}{k_{II}} = \dfrac{p_I}{p_{II}} = \dfrac{v_I}{v_{II}} = 2$.

$$T = \frac{4k_I k_{II}}{(k_I + k_{II})^2} = \frac{4(k_I/k_{II})}{\left(\frac{k_I}{k_{II}}+1\right)^2} = \frac{4\cdot 2}{3^2} = \frac{8}{9}$$

So we'll see $\boxed{\frac{8}{9}\,mA \text{ Transmitted}}$

and $\boxed{\frac{1}{9}\,mA}$ Reflected.

2] $\psi = C e^{-\frac{m\omega}{2\hbar}x^2}$, $C = \left(\dfrac{m\omega}{\pi\hbar}\right)^{1/4}$.

a] $\langle x \rangle = 0$ by symmetry.

b] $$\langle x^2 \rangle = C^2 \int_{-\infty}^{\infty} dx\, x^2 e^{-\frac{m\omega}{\hbar}x^2} = C^2 \frac{\pi^{1/2}}{2}\left(\frac{\hbar^{3/2}}{m^{3/2}\omega^{3/2}}\right)$$

$$= \frac{m^{1/2}\omega^{1/2}}{\pi^{1/2}\hbar^{1/2}} \cdot \frac{\pi^{1/2}}{2} \cdot \frac{\hbar^{3/2}}{m^{3/2}\omega^{3/2}} = \boxed{\frac{1}{2}\frac{\hbar}{m\omega}}$$

c] $$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \boxed{\left(\frac{\hbar}{2m\omega}\right)^{1/2}}$$

d] $\langle p \rangle = 0$

e] $E = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$, so

$$\frac{p^2}{2m} = E - \frac{1}{2}m\omega^2 x^2$$

$$\Rightarrow \langle p^2 \rangle = 2m\langle E \rangle - \frac{1}{2}\cdot 2m\, m\omega^2 \langle x^2 \rangle$$

$$= \cancel{2}m \cdot \frac{1}{\cancel{2}}\hbar\omega - m^2\omega^2 \frac{\hbar}{2m\omega}$$

$$= \boxed{\frac{1}{2}\hbar m\omega}$$

~~d~~ f] $\Delta p = \left(\frac{1}{2}\hbar m\omega\right)^{1/2}$

g] $\Delta x \Delta p = \left(\frac{1}{2}\frac{\hbar}{m\omega}\right)^{1/2}\left(\frac{1}{2}\hbar m\omega\right)^{1/2} = \boxed{\frac{\hbar}{2}}$

The minimum uncertainty bound!!