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Quantum Universe (4E)
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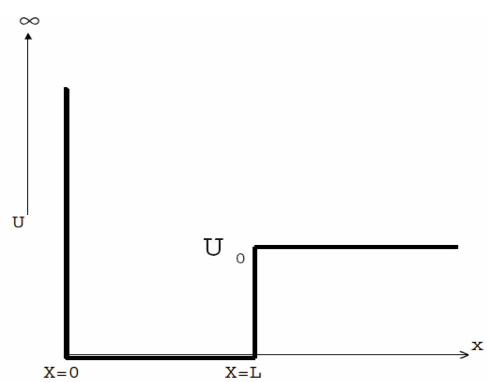
Quiz # 2 (April 23 2004)

Problem 1 : Particles Under Strong Interaction Force [8 pts]

A particle with mass m moves in a potential U(x) = A|x| , where A is a positive constant. In a simplified picture, quarks (constituents of protons and neutrons) have a potential energy of interaction of approximately this form, where x represents the separation between a pair of quarks. Because U(x)$\rightarrow\infty$ as x$\rightarrow\infty$, it is not possible to separate quarks from each other (a phenomenon called Quark confinement). (a) Classically, what is the force acting on this particle as a function of x ? (b) Using the uncertainty relation determine the minimum energy of the particle. **Hint:** First find the expression for the total energy of the system as just a function of x.

Problem 2: The Case Of A Hybrid Square Well [12 pts]

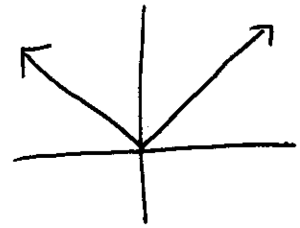
Consider a square well having an infinite wall at x=0 and a wall of height U_0 at x=L. For the case E<U_0, (a) obtain physical solutions of the Schrodinger equation inside the well ($0 \le x \le L$) and (b) in the region beyond (x>L) that satisfy the appropriate boundary conditions at x = 0 and x = ∞. (c) Sketch the wavefunction (d) Enforce the proper matching condition at x=L to show that $kL/\sin kL = \sqrt{\dfrac{2mU_0L^2}{\hbar^2}}$ (e) Are there conditions for which no solution is possible? Explain.



Hint: For (d) write $k^2 = \dfrac{2mE}{\hbar^2}$; $\alpha^2 = 2m(U_0 - E)/\hbar^2$, $k^2 + \alpha^2 = \dfrac{2mU_0}{\hbar^2}$ and use $1+\cot^2\theta = \csc^2\theta$.

Phys 4E Quiz 2

1a $F = -\dfrac{\partial U}{\partial x} = -\dfrac{\partial}{\partial x}\begin{cases} Ax & x>0 \\ -Ax & x<0 \end{cases}$



$$= \begin{cases} -A & x>0 \\ A & x<0 \end{cases}$$

Makes sense – always pushes back to $x=0$.

1b $E = \dfrac{p^2}{2m} + A|x|.$

Take $p \approx \cancel{\langle p \rangle}^{\,0} + \Delta p = \dfrac{\hbar}{2\Delta x} = \dfrac{\hbar}{2x}$, since $x \approx \cancel{\langle x \rangle}^{\,0} + \Delta x$.

$$\therefore E = \frac{\hbar^2}{8mx^2} + A|x|$$

$$\frac{\partial E}{\partial x} = \frac{-\hbar^2}{4mx^3} \pm A = 0 \Rightarrow \quad x = \pm\left(\frac{\hbar^2}{4mA}\right)^{1/3}$$

So $E_{min} = \dfrac{\hbar^2}{8m}\left(\dfrac{4mA}{\hbar^2}\right)^{2/3} + A\left(\dfrac{\hbar^2}{4mA}\right)^{1/3}$

$$= \frac{1}{2}\left[\left(\frac{\hbar^2A^2}{4m}\right)^{1/3}\right] + \left(\frac{\hbar^2A^2}{4m\cancel{A}}\right)^{1/3} = \boxed{\frac{3}{2}\left(\frac{\hbar^2A^2}{4m}\right)^{1/3}}$$

2a] For $0 \le x \le L$, $V=0$ so the Sch. Equ. is

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} = E\psi \Rightarrow \psi(x) = A\sin(kx) + B\cos(kx)$$

where $k^2 = \frac{2mE}{\hbar^2}$.

So since $V=\infty$ at $x=0$, we must have $\psi(0)=0$

$\Rightarrow B=0$.

$\therefore \boxed{\psi(x) = A\sin(kx)}$ for $0 \le x \le L$.

b] Now $V=U_0$, so Sch. Eqn. is

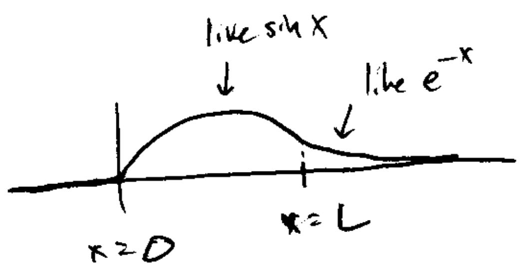
$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + U_0\psi = E\psi \Rightarrow \frac{\partial^2\psi}{\partial x^2} = \frac{2m(U_0-E)}{\hbar^2}\psi$$

$$\Rightarrow \psi(x) = Ce^{-\alpha x} + De^{\alpha x}, \quad \alpha^2 = \frac{2m(U_0-E)}{\hbar^2}.$$

But at $x=\infty$, $e^{\alpha x}$ blows up $\Rightarrow D=0$

$\boxed{\psi(x) = Ce^{-\alpha x}}$ for $x \ge L$.

c] Ground state:



d] ψ matches @ $x=L$: $A\sin(kL) = Ce^{-\alpha L}$

ψ' matches @ $x=L$: $kA\cos(kL) = -C\alpha e^{-\alpha L}$

Divide bottom eqn. by top: $k\cot(kL) = -\alpha$

Square both sides: $k^2\cot^2(kL) = \alpha^2$

But $\alpha^2 = \dfrac{2m(U_0-E)}{\hbar^2} = \dfrac{2mU_0}{\hbar^2} - k^2$

So $\cot^2(kL) = \dfrac{\alpha^2}{k^2} = \dfrac{2mU_0}{\hbar^2 k^2} - 1$

$$\Rightarrow \quad 1+\cot^2(kL) = \frac{2mU_0}{\hbar^2 k^2}$$

$$\Rightarrow \quad \csc^2(kL) = \frac{2mU_0}{\hbar^2 k^2}$$

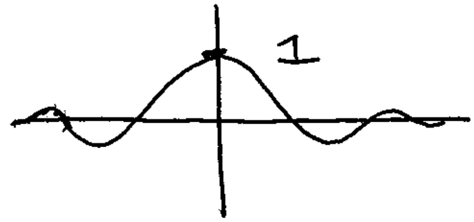
$$\Rightarrow \quad \sin^2(kL) = \frac{\hbar^2 k^2}{2mU_0}$$

$$\Rightarrow \quad \frac{\sin^2(kL)}{k^2L^2} = \frac{\hbar^2}{2mU_0L^2}$$

$$\Rightarrow \quad \frac{\sin(kL)}{kL} = \sqrt{\frac{\hbar^2}{2mU_0L^2}}$$

$$\Rightarrow \quad \boxed{\frac{kL}{\sin(kL)} = \sqrt{\frac{2mU_0L^2}{\hbar^2}}}$$

e] $\frac{\sin x}{x}$ looks like



So $\frac{\sin x}{x} \leq 1$

$\therefore$ No solns. if $\boxed{\sqrt{\frac{\hbar^2}{2mV_0L^2}} > 1}$

aka $\sqrt{\frac{2mV_0L^2}{\hbar^2}} < 1$