

Phys 4E Week 5 HW

Ch 6: 28, 29, 30, 31, 36, 37, 41, 42, 56, 58, 61, 62

For 28-31 see last week's solutions.

36 a $\psi_1(x) = C_1 x e^{-\frac{m\omega x^2}{2\hbar}}$

So $\int_{-\infty}^{\infty} C_1^2 x^2 e^{-m\omega x^2/\hbar} dx = 1$. But $\int_{-\infty}^{\infty} x^2 e^{-ax^2} dx = \frac{1}{2} \sqrt{\frac{\pi}{a^3}}$

So $\int_{-\infty}^{\infty} x^2 e^{-m\omega x^2/\hbar} dx = \frac{1}{2} \sqrt{\pi} \left(\frac{\hbar}{m\omega}\right)^{3/2} = \left(\frac{\pi \hbar^3}{4m^3\omega^3}\right)^{1/2}$

$\therefore C_1^2 \left(\frac{4m^3\omega^3}{\pi \hbar^3}\right)^{1/2} \Rightarrow C_1 = \left(\frac{4m^3\omega^3}{\pi \hbar^3}\right)^{1/4}$

b $\langle x \rangle = \int_{-\infty}^{\infty} C_1^2 x^3 e^{-m\omega x^2/\hbar} dx = 0$ b/c $x^3 e^{-x^2}$ is odd.
 ($f(-x) = -f(x)$)

c $\langle x^2 \rangle = \int_{-\infty}^{\infty} C_1^2 x^4 e^{-m\omega x^2/\hbar} dx = C_1^2 \int_{-\infty}^{\infty} x^4 e^{-m\omega x^2/\hbar} dx$

But $\int_{-\infty}^{\infty} x^4 e^{-ax^2} dx = \frac{3}{4} \sqrt{\frac{\pi}{a^5}}$, so $\langle x^2 \rangle = C_1^2 \frac{3}{4} \sqrt{\pi} \left(\frac{\hbar}{m\omega}\right)^{5/2}$

$\therefore \langle x^2 \rangle = 2 \left(\frac{m^3\omega^3}{\pi \hbar^3}\right)^{1/2} \cdot \frac{3}{4} \sqrt{\pi} \left(\frac{\hbar^5}{m^5\omega^5}\right)^{1/2} = \boxed{\frac{3}{2} \frac{\hbar}{m\omega}}$

$$2] \quad V(x) = \frac{1}{2} m \omega^2 x^2, \text{ so } \langle V(x) \rangle = \frac{1}{2} m \omega^2 \langle x^2 \rangle$$

$$= \frac{1}{2} m \omega^2 \cdot \frac{3}{2} \frac{\hbar}{m \omega} = \boxed{\frac{3}{4} \hbar \omega}$$

$$37] \quad \Delta p = \frac{\hbar}{2 \Delta x} = \boxed{\frac{\hbar}{4A}}$$

$$b] \quad \Rightarrow KE = \frac{p^2}{2m} = \frac{\hbar^2}{32mA^2}$$

$$\text{But } E = \frac{1}{2} k A^2 = \frac{1}{2} \hbar \omega \Rightarrow A^2 = \frac{\hbar \omega}{k}$$

$$\text{and since } \omega = \sqrt{\frac{k}{m}}, \quad k = m \omega^2 \Rightarrow A^2 = \frac{\hbar \omega}{m \omega^2} = \frac{\hbar}{m \omega}$$

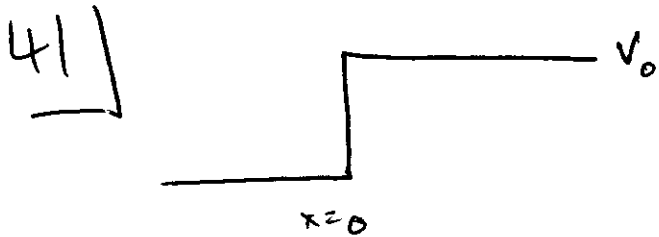
$$\Rightarrow KE = \frac{\hbar^2 m \omega}{32m \hbar} = \frac{1}{32} \hbar \omega$$

The ground state total energy is $\frac{1}{2} \hbar \omega$, so the H.U.P. estimate is off by a factor of 16.

$\langle KE \rangle = \langle E \rangle - \langle V \rangle$. For the ground state,

$$\langle V \rangle = \frac{1}{2} m \omega^2 \langle x^2 \rangle = \frac{1}{2} m \omega^2 \cdot \frac{1}{2} \frac{\hbar}{m \omega} = \frac{1}{4} \hbar \omega \text{ (similar to Prob. 36)}$$

$\Rightarrow \langle KE \rangle = \frac{1}{2} \hbar \omega - \frac{1}{4} \hbar \omega = \frac{1}{4} \hbar \omega$. So we're off by a factor of 8.



a] Sch. Egn. is $-\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V_0 \psi = E \psi$ for $x > 0$

$$\Rightarrow \frac{\partial^2 \psi}{\partial x^2} = -\frac{2m(E-V_0)}{\hbar^2} \psi$$

$$\text{So } k_2^2 = \frac{2m(E-V_0)}{\hbar^2} = \frac{2mV_0}{\hbar^2}$$

$$\text{But } k_1^2 = \frac{2m \cdot 2V_0}{\hbar^2} = \frac{4mV_0}{\hbar^2} \Rightarrow k_2^2 = \frac{k_1^2}{2} \Rightarrow \boxed{k_2 = \frac{k_1}{\sqrt{2}}}$$

b] For $x < 0$, $\psi_I = A e^{ik_1 x} + B e^{-ik_1 x}$
 $x > 0$, $\psi_{II} = C e^{ik_2 x}$

Matching: $\psi_I(0) = \psi_{II}(0): A + B = C$

$$\psi_I'(0) = \psi_{II}'(0): k_1(A - B) = k_2 C$$

$$R = \frac{|B|^2}{|A|^2} = \left(\frac{k_1 - k_2}{k_1 + k_2} \right)^2 = \left(\frac{\sqrt{2} - 1}{\sqrt{2} + 1} \right)^2 = \boxed{2.94\%}$$

c] $T = 1 - R = \boxed{97.06\%}$

$$Q] T(1,000,000) = \boxed{970,600}$$

Classically, they all would keep going.

42]

$$a] \text{ Now } k_2^2 = \frac{2m(2V_0 + V_0)}{\hbar^2} = \frac{6mV_0}{\hbar^2}$$

$$\text{So } k_2^2 = \frac{3}{2}k_1^2 \Rightarrow \boxed{k_2 = \sqrt{\frac{3}{2}}k_1}$$

$$b] R = \left(\frac{k_1 - k_2}{k_1 + k_2} \right)^2 = \left(\frac{1 - \sqrt{3/2}}{1 + \sqrt{3/2}} \right)^2 = \boxed{1.02\%}$$

$$c] T = 1 - R = \boxed{98.98\%}$$

Q] $\boxed{989,800}$. Classically, they all keep going.

$$56] a] T \approx 16 \frac{E}{V_0} \left(1 - \frac{E}{V_0}\right) e^{-2\alpha a}$$

$$E = 10 \text{ eV}, V_0 = 25 \text{ eV}$$

~~$$\alpha = \left[\frac{2m(V_0 - E)}{\hbar^2} \right]^{1/2}$$~~

$$\alpha = \left[\frac{2m(V_0 - E)}{\hbar^2} \right]^{1/2}, a = 1 \text{ nm.}$$

$$\text{So } \alpha a = \left[\frac{2mc^2(V_0 - E)}{(\hbar c)^2} \right]^{1/2} a = \left[2(511 \times 10^3 \text{ eV})(15 \text{ eV}) \right]^{1/2} \frac{a}{197.33 \text{ eV} \cdot \text{nm}}$$

$$S_0 \alpha a = (19.8) \frac{a}{\text{nm}}$$

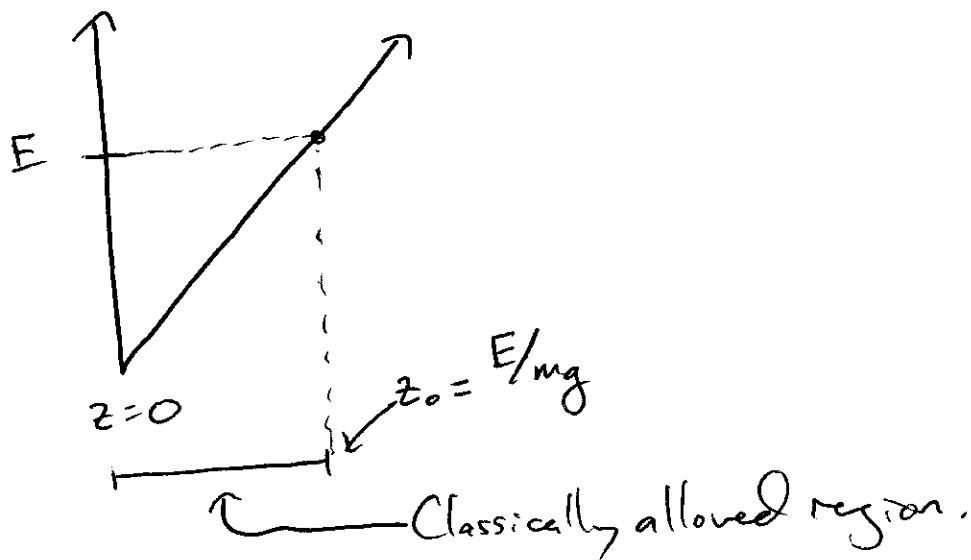
$$16 \left(\frac{E}{V_0} \right) \left(1 - \frac{E}{V_0} \right) = 3.84$$

$$S_0 T = 3.84 e^{-39.6a} = \boxed{2.5 \times 10^{-16}}$$

b) For $a = 0.1 \text{ nm}$,

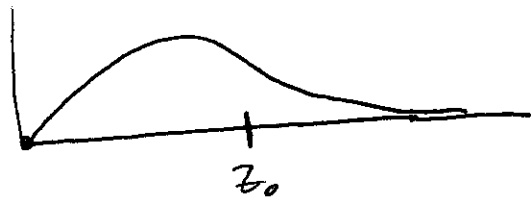
$$T = 3.84 e^{-3.96} = \boxed{7.3 \times 10^{-2}} \quad \underline{\text{Much bigger!!}}$$

58

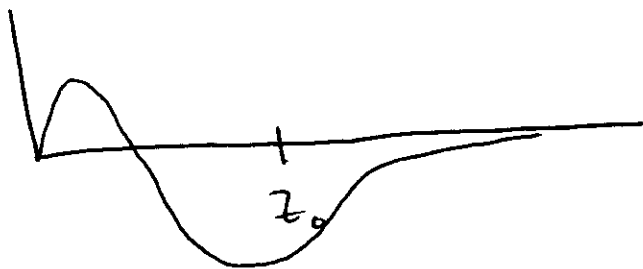


As z approaches the disallowed region, the "wavenumber" will decrease: Treating $k \hat{=} \left(\frac{2m(E-V)}{\hbar^2} \right)$, as V increases, k decreases $\rightarrow \lambda$ increases:

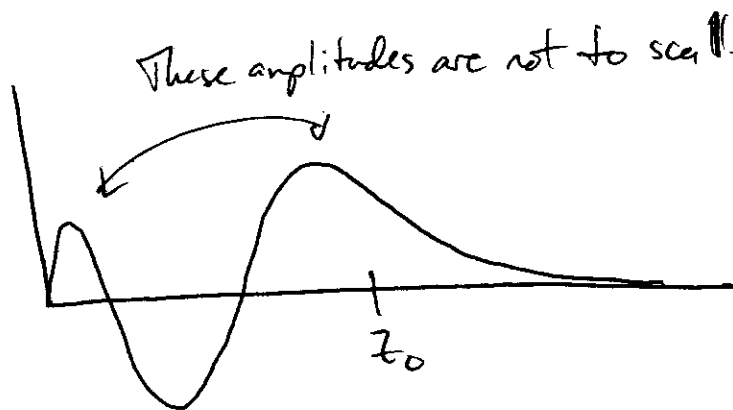
Ground state: One bump in well, Then decay:



1st excited state: 2 bumps, λ increasing as $z \rightarrow z_0$, Then decay.



2nd excited state: 3 bumps, Then decay:



61

6-74 sez

$$\psi_{\text{I}} = Ae^{ik_1x} + Be^{-ik_1x}$$

$$\psi_{\text{II}} = Ce^{-\alpha x} + De^{\alpha x}$$

$$\psi_{\text{III}} = Fe^{ik_1x} + Ge^{-ik_1x}$$

Set $G=0$ for incident from left.

$$\therefore \text{Matching: } \psi_{\text{I}}(0) = \psi_{\text{II}}(0): A + B = C + D \quad (1)$$

$$\psi_{\text{II}}(a) = \psi_{\text{III}}(a): C e^{-\alpha a} + D e^{\alpha a} = F e^{ik_1 a} \quad (2)$$

$$\psi'_{\text{I}}(0) = \psi'_{\text{II}}(0): k_1(A - B) = \alpha(-C + D) \quad (3)$$

$$\psi'_{\text{II}}(a) = \psi'_{\text{III}}(a): \alpha(-C e^{-\alpha a} + D e^{\alpha a}) = +ik_1 F e^{ik_1 a} \quad (4)$$

5 unknowns, 4 eqns. Now solve them with the power of algebra, and it gives the answer (I'm not gonna do it).

$$T = \left[1 + \frac{\sinh^2 \alpha a}{4 \frac{E}{v_0} \left(1 - \frac{E}{v_0}\right)} \right]^{-1}$$

$$b) \text{ If } \alpha a \gg 1, \sinh^2 \alpha a = \left(\frac{e^{+\alpha a} - e^{-\alpha a}}{2} \right)^2 \approx \left(\frac{e^{\alpha a}}{2} \right)^2 = \frac{e^{2\alpha a}}{4}$$

$$S_0 T = \left[1 + \frac{e^{2\alpha a}}{16 \frac{E}{v_0} \left(1 - \frac{E}{v_0}\right)} \right]^{-1} = 16 \frac{E}{v_0} \left(1 - \frac{E}{v_0}\right) e^{-2\alpha a} \frac{1}{1 + \frac{16 \frac{E}{v_0} \left(1 - \frac{E}{v_0}\right)}{e^{2\alpha a}}}$$

$$\approx \boxed{16 \frac{E}{v_0} \left(1 - \frac{E}{v_0}\right) e^{-2\alpha a}}$$

$$62] \quad \psi_{II} = C e^{-\alpha x}; \quad \alpha^2 = 2m(V_0 - E)/\hbar^2$$

$$\psi_{II} = A e^{ikx} + B e^{-ikx}; \quad k^2 = 2mE/\hbar^2$$

Matching: $A + B = C, \quad ik(A - B) = -\alpha C.$

$$\Rightarrow A - B = \frac{i\alpha}{k} C$$

$$\Rightarrow 2A = \left(1 + \frac{i\alpha}{k}\right) C$$

$$\Rightarrow C = \frac{2A}{\left(1 + \frac{i\alpha}{k}\right)} \Rightarrow |C|^2 = \frac{4|A|^2}{\left(1 + \frac{\alpha^2}{k^2}\right)}$$

$$= \frac{4|A|^2}{1 + \frac{V_0 - E}{E}}$$

$$= \frac{4}{1 + \frac{V_0 - E}{E}} \quad (|A|^2 = 1).$$

So $\psi_{II} = 2 e^{-\alpha x} \Rightarrow |\psi|^2 = 4 e^{-2\alpha x}$

$$\text{But } \alpha = \left(\frac{2mc^2 \cdot 20\text{eV}}{\hbar^2 c^2}\right)^{1/2} = \left(\frac{2.938\text{MeV} \cdot 20\text{eV}}{(197.3\text{MeV}\cdot\text{fm})^2}\right)^{1/2} = \frac{9.8 \times 10^{-4}}{\text{fm}}$$

Graph: $x=1 \text{ fm}$: $141^2 = 3.99$

2 fm 3.984

3 fm 3.976

4 fm 3.968

5 fm 3.960

