A PhD Thesis Fit For a Prince

• Matter Wave!
  – “Pilot wave” of $\lambda = h/p = h/(\gamma mv)$
  – frequency $f = E/h$

• Consequence:
  – If matter has wave like properties then there would be interference (destructive & constructive)
    • Use analogy of standing waves on a plucked string to explain the quantization condition of Bohr orbits
De Broglie’s Explanation of Bohr’s Quantization

Standing waves in H atom:
Constructive interference when
\[ n\lambda = 2\pi r \]

since \[ \lambda = \frac{h}{p} = \frac{h}{mv} \] .....(NR)

\[ \Rightarrow \frac{nh}{mv} = 2\pi r \]

\[ \Rightarrow nh = mvr \]

Angular momentum
Quantization condition!

This is too intense! Must verify such “loony tunes” with experiment

Just What is Waving in Matter Waves?

For waves in an ocean, it’s the water that “waves”
For sound waves, it’s the molecules in medium
For light it’s the E & B vectors that oscillate

• What’s “waving” for matter waves?
  – It’s the PROBABILITY OF FINDING THE PARTICLE that waves!
  – Particle can be represented by a wave packet
    • At a certain location (x)
    • At a certain time (t)
    • Made by superposition of many sinusoidal waves of different amplitudes, wavelengths \( \lambda \) and frequency \( f \)
  – It’s a “pulse” of probability in spacetime
What Wave Does Not Describe a Particle

- What wave form can be associated with particle’s pilot wave?
- A traveling sinusoidal wave? \( y = A \cos (kx - \omega t + \Phi) \)
- Since de Broglie “pilot wave” represents particle, it must travel with same speed as particle …… (like me and my shadow)

\[
y = A \cos (kx - \omega t + \Phi)
\]

\[
k = \frac{2\pi}{\lambda}, \quad w = 2\pi f
\]

In Matter:

\[
\lambda = \frac{h}{p} = \frac{h}{\gamma m v}
\]

Conflicts with Relativity → Unphysical

Need “wave packets” localized Spatially (x) and Temporally (t)

Wave Group or Wave Pulse

- Wave Group/packet:
  - Superposition of many sinusoidal waves with different wavelengths and frequencies
  - Localized in space, time
  - Size designated by
    - \( \Delta x \) or \( \Delta t \)
  - Wave groups travel with the speed \( v_p = \frac{\lambda}{f} \)
- Constructing Wave Packets
  - Add waves of diff \( \lambda \),
  - For each wave, pick
    - Amplitude
    - Phase
  - Constructive interference over the space-time of particle
  - Destructive interference elsewhere!

Imagine Wave pulse moving along a string: its localized in time and space (unlike a pure harmonic wave)
How To Make Wave Packets: Just Beat it!

- Superposition of two sound waves of slightly different frequencies $f_1$ and $f_2$, $f_1 \neq f_2$
- Pattern of beats is a series of wave packets
- Beat frequency $f_{\text{beat}} = f_2 - f_1 = \Delta f$
- $\Delta f$ = range of frequencies that are superimposed to form the wave packet

Resulting wave’s "displacement" $y = y_1 + y_2$

$y = A \left[ \cos(k_1 x - w_1 t) + \cos(k_2 x - w_2 t) \right]$

Trigonometry: $\cos A + \cos B = 2 \cos \left( \frac{A+B}{2} \right) \cos \left( \frac{A-B}{2} \right)$

$\therefore y = 2A \left[ \left( \cos \left( \frac{k_1 - k_2}{2} x - \frac{w_2 - w_1}{2} t \right) \right) \cos \left( \frac{k_1 + k_2}{2} x - \frac{w_1 + w_2}{2} t \right) \right]$

since $k_2 \equiv k_1 \equiv k_{\text{ave}}$, $w_2 \equiv w_1 \equiv w_{\text{ave}}$, $\Delta k \ll k$, $\Delta w \ll w$

$\therefore y = 2A \left[ \cos \left( \frac{\Delta k}{2} x - \frac{\Delta w}{2} t \right) \right] = y = A \cdot \cos(kx - wt)$, $A'$ oscillates in $x,t$

$A' = 2A \left( \cos \left( \frac{\Delta k}{2} x - \frac{\Delta w}{2} t \right) \right) = \text{modulated amplitude}$

Phase Vel $V_p = \frac{w_{\text{ave}}}{k_{\text{ave}}}$

Group Vel $V_g = \frac{\Delta w}{\Delta k}$

$V_g : \text{Vel of envelope} = \frac{dw}{dk}$

Addition of 2 Waves with slightly different wavelengths and slightly different frequencies.
Non-repeating wave packet can be created thru superposition of many waves of similar (but different) frequencies and wavelengths.

Wave Packet: Localization

- Finite # of diff. Monochromatic waves always produce INFINITE sequence of repeating wave groups \( \rightarrow \) can‘t describe (localized) particle
- To make localized wave packet, add “infinite” # of waves with

Well chosen Amplitudes \( A \), Waves\# \( k \), ang.

\[ \psi(x,t) = \int A(k) e^{i(kx-wt)} dk \]

\( A(k) = \) Amplitude Fn
\( \Rightarrow \) diff waves of diff \( k \)
\( w = w(k) \), depends on type of wave, media

Group Velocity \( V_g = \frac{d\omega}{dk} \)

Gaussian Function

- Gaussian distribution of some quantity with\( x \) is a form is
In a Wave Packet: \( w = w(k) \)

Group Velocity \( V_g = \frac{dw}{dk} \)

Since \( V_p = wk \) (def) \( \Rightarrow w = kV_p \)

\[ V_g = \frac{dw}{dk} = V_p k \frac{dk}{dw} + k \frac{dV_p}{dk} \]

usually \( V_p = V_p(k \ or \lambda) \)

Material in which \( V_p \) varies with \( \lambda \) are said to be Dispersive

Individual harmonic waves making a wave pulse travel at different \( V_p \) thus changing shape of pulse and become spread out

In non-dispersive media, \( V_g = V_p \)

In dispersive media \( V_g \neq V_p \), depends on \( \frac{dV_p}{dk} \)

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Consider An Electron:

- mass = \( m \)
- velocity = \( v \)
- momentum = \( p \)
- Energy \( E = \hbar \omega = \gamma mc^2 \)
- \( \omega = 2\pi f = \frac{2\pi}{\hbar} \gamma mc^2 \)
- Wavelength \( \lambda = \frac{\hbar}{p} \)
- \( k = \frac{2\pi}{\lambda} \Rightarrow k = \frac{2\pi}{\hbar} \gamma mv \)

Group Velocity: \( V_g = \frac{dw}{dk} = \frac{dv}{dk} / \frac{dv}{dw} \)

\[ \frac{dw}{dv} = \frac{d}{dv} \left[ \frac{2\pi \hbar}{\hbar^2} \left( \frac{v}{c} \right)^2 \right] = \frac{2\pi m}{h(1-\left( \frac{v}{c} \right)^2)^{3/2}} \] \& \[ \frac{dk}{dv} = \frac{d}{dv} \left[ \frac{2\pi}{h(1-\left( \frac{v}{c} \right)^2)^{3/2}} \right] = \frac{2\pi m}{h(1-\left( \frac{v}{c} \right)^2)^{3/2}} \]

\[ V_g = \frac{dw}{dk} = \frac{dw}{dv} / \frac{dv}{dw} = v \Rightarrow \text{Group velocity of electron Wave packet "pilot wave" is same as electron's physical velocity} \]

But velocity of individual waves making up the wave packet \( V_p = \frac{w}{k} = \frac{c^2}{v} > c! \) (not physical)
Wave Packets & Uncertainty Principles

- Distance $\Delta X$ between adjacent minima = $(X_2)_{\text{node}} - (X_1)_{\text{node}}$
- Define $X_1=0$ then phase diff from $X_1 \rightarrow X_2 = \pi$ (similarly for $t_1 \rightarrow t_2$)

What can we learn from this simple model?

Node at $y = 0 = 2A \cos \left( \frac{\Delta w}{2} \frac{x - \Delta k}{2} \right)$, Examine $x$ or $t$ behavior

$\Rightarrow$ in $x$: $\Delta k \Delta x = \pi$ $\Rightarrow$ Need to combine many waves of diff. $k$ to make small $\Delta x$ pulse

$\Delta x = \frac{\pi}{\Delta k}$, for small $\Delta x \rightarrow 0 \Rightarrow \Delta k \rightarrow \infty$ & Vice Versa

$\Delta w / \Delta \omega = \pi$ $\Rightarrow$ Need to combine many waves of diff $\omega$ to make small $\Delta t$ pulse

$\Delta t = \frac{\pi}{\Delta \omega}$, for small $\Delta t \rightarrow 0 \Rightarrow \Delta \omega \rightarrow \infty$ & Vice Versa

Signal Transmission and Bandwidth Theory

- Short duration pulses are used to transmit digital info
  - Over phone line as brief tone pulses
  - Over satellite link as brief radio pulses
  - Over optical fiber as brief laser light pulses
- Regardless of type of wave or medium, any wave pulse must obey the fundamental relation
  \[ \Delta \omega \Delta t \approx \pi \]
- Range of frequencies that can be transmitted are called bandwidth of the medium
- Shortest possible pulse that can be transmitted through a medium is $\Delta t_{\text{min}} \approx \pi / \Delta \omega$
- Higher bandwidths transmit shorter pulses & allows high data rate

We added two Sinusoidal waves

$y = 2A \left[ \cos \left( \frac{\Delta k}{2} x - \frac{\Delta w}{2} t \right) \cos (kx - \omega t) \right]$
Wave Packets & Uncertainty Principles of Subatomic Physics

in space $x$: $\Delta k \cdot \Delta x = \pi$  $\Rightarrow$  since $k = \frac{2\pi}{\lambda}$, $p = \frac{h}{\lambda}$

$\Rightarrow \Delta p \cdot \Delta x = \frac{h}{2}$

usually one writes $\Delta p \cdot \Delta x \geq \frac{\hbar}{2}$ approximate relation

In time $t$: $\Delta \omega \cdot \Delta t = \pi$  $\Rightarrow$  since $\omega = 2\pi f$, $E = hf$

$\Rightarrow \Delta E \cdot \Delta t = \frac{\hbar}{2}$

usually one writes $\Delta E \cdot \Delta t \geq \frac{\hbar}{2}$ approximate relation

What do these inequalities mean physically?

Know the Error of Thy Ways: Measurement Error $\rightarrow \Delta$

- Measurements are made by observing something: length, time, momentum, energy
- All measurements have some (limited) precision...no matter the instrument used
- Examples:
  - How long is a desk? $L = (5 \pm 0.1)$ m = $L \pm \Delta L$ (depends on ruler used)
  - How long was this lecture? $T = (50 \pm 1)$ minutes = $T \pm \Delta T$ (depends on the accuracy of your watch)
  - How much does Prof. Sharma weigh? $M = (1000 \pm 700)$ kg = $m \pm \Delta m$
    - Is this a correct measure of my weight?
      - Correct (because of large error reported) but imprecise
      - My correct weight is covered by the (large) error in observation

Length Measure

Voltage (or time) Measure
Measurement Error : $x \pm \Delta x$

- Measurement errors are unavoidable since the measurement procedure is an experimental one.
- True value of a measurable quantity is an abstract concept.
- In a set of repeated measurements with random errors, the distribution of measurements resembles a Gaussian distribution characterized by the parameter $\sigma$ or $\Delta$ characterizing the width of the distribution.

**The Gauss, or Normal, Distribution**

$$G_{x,\sigma}(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-(x - \bar{x})^2/2\sigma^2}.$$

**Interpreting Measurements with random Error : $\Delta$**

*Figure 5.12.* The shaded area between $X \pm t\sigma$ is the probability of a measurement within $t$ standard deviations of $X$. 

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<th>0</th>
<th>0.25</th>
<th>0.5</th>
<th>0.75</th>
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<table>
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<th>1.645</th>
<th>2.326</th>
<th>2.576</th>
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<td>0.997</td>
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<td>0.9998</td>
<td>0.99994</td>
<td>0.99999</td>
<td>0.999999</td>
</tr>
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</table>
Where in the World is Carmen San Diego?

- Carmen San Diego hidden inside a big box of length L
- Suppose you can’t see thru the (blue) box, what is you best estimate of her location inside box (she could be anywhere inside the box)

Your best unbiased measure would be $x = L/2 \pm L/2$

There is no perfect measurement, there are always measurement error

Wave Packets & Matter Waves

What is the Wave Length of this wave packet?

$\lambda - \Delta \lambda < \lambda < \lambda + \Delta \lambda$

De Broglie wavelength $\lambda = h/p$

$\rightarrow$ Momentum Uncertainty: $p - \Delta p < p < p + \Delta p$

Similarly for frequency $\omega$ or $f$

$\omega - \Delta \omega < \omega < \omega + \Delta \omega$

Planck’s condition $E = hf = h\omega/2$

$\rightarrow$ $E - \Delta E < E < E + \Delta E$
Back to Heisenberg's Uncertainty Principle & Δ

• Δx. Δp ≥ h/4π ⇒
  - If the measurement of the position of a particle is made with a precision Δx and a SIMULTANEOUS measurement of its momentum p in the X direction, then the product of the two uncertainties (measurement errors) can never be smaller than ≅h/4π irrespective of how precise the measurement tools.

• ΔE. Δt ≥ h/4π ⇒
  - If the measurement of the energy E of a particle is made with a precision ΔE and it took time Δt to make that measurement, then the product of the two uncertainties (measurement errors) can never be smaller than ≅h/4π irrespective of how precise the measurement tools.

These rules arise from the way we constructed the Wave packets describing Matter “pilot” waves.

The Act of Observation (Compton Scattering)

Observations of particle motion by means of scattered illumination. When the incident wavelength is reduced to accommodate the size of the particle, the momentum transferred by the photon becomes large enough to disturb the observed motion.

Visible light illuminating a macroscopic object.

Act of observation disturbs the observed system.

X ray illuminating an atomic electron.
Compton Scattering: Shining light to observe electron

\[ \lambda = \frac{h}{p} = \frac{hc}{E} = \frac{c}{f} \]

The act of Observation DISTURBS the object being watched, here the electron moves away from where it was originally.

Act of Watching: A Thought Experiment

Before collision

Incident photon

Electron

After collision

Scattered photon

Recoiling electron

Observed Diffraction pattern

\[ p = \frac{h}{\lambda} \]

\[ \epsilon \text{ initially at rest} \]

\[ \Delta x \]

\[ f_0, \lambda_0 \]
Diffraction By a Circular Aperture (Lens)

See Resnick, Halliday Walker 6th Ed (on S.Reserve), Ch 37, pages 898-900

Diffraction pattern of a point source of light thru a lens (circular aperture of size d)

First minimum of diffraction pattern is located by

\[ \sin \theta = 1.22 \frac{\lambda}{d} \]

See previous picture for definitions of \( \theta, \lambda, d \)

Resolving Power of Light Thru a Lens

Image of 2 separate point sources formed by a converging lens of diameter d, ability to resolve them depends on \( \lambda \) & d because of the inherent diffraction in image formation

Resolving power \( \Delta x \approx \frac{\lambda}{2 \sin \theta} \)

\( \theta \) Depends on d
Putting it all together: act of Observing an electron

- Incident light \((p, \lambda)\) scatters off electron
- To be collected by lens, \(\gamma\) must scatter thru angle \(\alpha\)
  \(-\theta \leq \alpha \leq \theta\)
- Due to Compton scatter, electron picks up momentum
  \[ P_x, P_y \leq \frac{h}{\lambda} \sin \theta \leq P_z = \frac{h}{\lambda} \sin \theta \]
  Electron momentum uncertainty is
  \[ \Delta p = \frac{2h}{\lambda} \sin \theta \]
- After passing thru lens, photon diffracts, lands somewhere on screen, image (of electron) is fuzzy
- How fuzzy? Optics says shortest distance between two resolvable points is:
  \[ \Delta x = \frac{\lambda}{2 \sin \theta} \]
- Larger the lens radius, larger the \(\theta\) \(\Rightarrow\) better resolution

Pseudo-Philosophical Aftermath of Uncertainty Principle

- Newtonian Physics & Deterministic physics topples over
  - Newton’s laws told you all you needed to know about trajectory of a particle
    - Apply a force, watch the particle go!
      - Know every thing! \(X, v, p, F, a\)
      - Can predict exact trajectory of particle if you had perfect device
  - No so in the subatomic world!
    - Of small momenta, forces, energies
    - Cant predict anything exactly
      - Can only predict probabilities
        - There is so much chance that the particle landed here or there
        - Cant be sure! ... cognizant of the errors of thy observations
  Philosophers went nuts! ... what has happened to nature
  Philosophers just talk, don’t do real life experiments!
All Measurements Have Associated Errors

- If your measuring apparatus has an intrinsic inaccuracy (error) of amount $\Delta p$
- Then results of measurement of momentum $p$ of an object at rest can easily yield a range of values accommodated by the measurement imprecision:
  - $-\Delta p \leq p \leq \Delta p$
- Similarly for all measurable quantities like $x$, $t$, Energy!

Matter Diffraction & Uncertainty Principle

Momentum measurement beyond slit show particle not moving exactly in Y direction, develops a X component Of motion $\Delta P_x = \frac{h}{2\pi a}$

[Diagram showing electron beam incident in Y direction, with probability distribution and diffraction pattern.]
Particle at Rest Between Two Walls

• Object of mass M at rest between two walls originally at infinity
• What happens to our perception of George as the walls are brought in?

On average, measure $\langle p \rangle = 0$
but there are quite large fluctuations!
Width of Distribution $= \Delta P$

$$\Delta P = \sqrt{(P^2)_{\text{ave}} - (P_{\text{ave}})^2}; \quad \Delta P \sim \frac{\hbar}{L}$$