Finite Potential Barrier

• There are no Infinite Potentials in the real world
  – Imagine the cost of an battery with infinite potential diff
    • Will cost infinite $ sum + not available at Radio Shack
  • Imagine a realistic potential: Large U compared to KE but not infinite

Classical Picture: A bound particle (no escape) in 0<x<L
Quantum Mechanical Picture: Use ΔEΔt ≤ ℏ/2π
Particle can leak out of the Box of finite potential P(|x|>L) ≠ 0
Finite Potential Well

\[-\hbar^2 \frac{d^2\psi(x)}{dx^2} + U\psi(x) = E\psi(x)\]

\[\Rightarrow \frac{d^2\psi(x)}{dx^2} = \frac{2m}{\hbar^2}(U - E)\psi(x)\]

\[= \alpha^2\psi(x); \quad \alpha = \sqrt{\frac{2m(U-E)}{\hbar^2}}\]

\[\Rightarrow \text{General Solutions: } \psi(x) = Ae^{\alpha x} + Be^{-\alpha x}\]

Require finiteness of \(\psi(x)\)

\[\Rightarrow \psi(x) = Ae^{\alpha x} \quad \text{.....} x<0 \quad (\text{region I})\]

\[\psi(x) = Ae^{-\alpha x} \quad \text{.....} x>L \quad (\text{region III})\]

Again, coefficients A & B come from matching conditions at the edge of the walls (x =0, L)

But note that wave fn at \(\psi(x)\) at (x =0, L) \(\neq 0\) !! (why?)

Further require Continuity of \(\psi(x)\) and \(\frac{d\psi(x)}{dx}\)

These lead to rather different wave functions

Finite Potential Well: Particle can Burrow Outside Box
Particle can be outside the box but only for a time $\Delta t = \frac{\hbar}{\Delta E}$.

$\Delta E$ is the energy the particle needs to borrow to get outside.

$\Delta E = U - E + KE$

The Cinderella act (of violating energy conservation) can't last very long.

Particle must hurry back (can't be caught with its hand inside the cookie-jar).

Penetration Length $\delta = \frac{1}{\alpha} = \frac{\hbar}{\sqrt{2m(U-E)}}$

If $U \gg E \Rightarrow$ Tiny penetration

If $U \rightarrow \infty \Rightarrow \delta \rightarrow 0$

$E_n = \frac{n^2 \pi^2 \hbar^2}{2m(L+2\delta)^2}$, $n = 1, 2, 3, 4...$

When $E = U$ then solutions blow up

$\Rightarrow$ Limits to number of bound states ($E_n < U$)

When $E > U$, the particle is not bound and can get either reflected or transmitted across the potential "barrier".
Simple Harmonic Oscillator: Quantum and Classical

Spring with Force Const $k$

Particle of mass $m$ within a potential $U(x)$

$F(x) = - \frac{dU(x)}{dx}$

$F(x=a) = \frac{dU(x)}{dx} = 0,$

$F(x=b) = 0,$ $F(x=c) = 0$ ...But...

look at the Curvature:

$\frac{d^2U}{dx^2} > 0$ (stable), $\frac{d^2U}{dx^2} < 0$ (unstable)

Stable Equilibrium: General Form:

$U(x) = U(a) + \frac{1}{2}k(x-a)^2$

Rescale $\Rightarrow U(x) = \frac{1}{2}k(x-a)^2$

Motion of a Classical Oscillator (ideal)

Ball originally displaced from its equilibrium position, motion confined between $x=0$ & $x=A$

$U(x) = \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2x^2; \omega = \sqrt{\frac{k}{m}} = \text{Ang. Freq}$

$E = \frac{1}{2}kA^2 \Rightarrow$ Changing $A$ changes $E$

$E$ can take any value & if $A \to 0$, $E \to 0$

Max. KE at $x = 0$, KE= 0 at $x=\pm A$
Quantum Picture: Harmonic Oscillator

Find the Ground state Wave Function $\psi(x)$

Find the Ground state Energy $E$ when $U(x) = \frac{1}{2} m \omega^2 x^2$

Time Dependent Schrodinger Eqn:

$$\frac{\hbar^2}{2m} \frac{d^2 \psi(x)}{dx^2} + \frac{1}{2} m \omega^2 x^2 \psi(x) = E \psi(x)$$

$$\Rightarrow \frac{d^2 \psi(x)}{dx^2} = \frac{2m}{\hbar^2} \left( E - \frac{1}{2} m \omega^2 x^2 \right) \psi(x) = 0$$

What $\psi(x)$ solves this?

Two guesses about the simplest Wavefunction:

1. $\psi(x)$ should be symmetric about $x$  
2. $\psi(x) \to 0$ as $x \to \infty$

$\psi(x)$ should be continuous & $\frac{d \psi(x)}{dx} = \text{continuous}$

My guess: $\psi(x) = C_0 e^{-\alpha x^2}$; Need to find $C_0$ & $\alpha$

What does this wavefunction & PDF look like?

Quantum Picture: Harmonic Oscillator

$P(x) = C^2_0 e^{-2\alpha x^2}$

$\psi(x) = C_0 e^{-\alpha x^2}$

How to Get $C_0$ & $\alpha$ ?? ... Try plugging in the wave-function into the time-independent Schr. Eqn.
Time Independent Sch. Eqn & The Harmonic Oscillator

Master Equation is: \[ \frac{\partial^2 \psi(x)}{\partial x^2} = \frac{2m}{\hbar^2} \left( \frac{1}{2} m \omega^2 x^2 - E \right) \psi(x) \]

Since \( \psi(x) = C_0 e^{-\alpha x^2} \), \( \frac{d\psi(x)}{dx} = C_0 (-2\alpha x) e^{-\alpha x^2} \),

\[ \frac{\partial^2 \psi(x)}{\partial x^2} = C_0 \frac{d(-2\alpha x)}{dx} e^{-\alpha x^2} + C_0 (-2\alpha x)^2 e^{-\alpha x^2} = C_0 [4\alpha^2 x^2 - 2\alpha] e^{-\alpha x^2} \]

\[ \Rightarrow C_0 [4\alpha^2 x^2 - 2\alpha] e^{-\alpha x^2} = \frac{2m}{\hbar^2} \left[ \frac{1}{2} m \omega^2 x^2 \right] - E C_0 e^{-\alpha x^2} \]

Match the coeff of \( x^2 \) and the Constant terms on LHS & RHS

\[ \Rightarrow 4\alpha^2 = \frac{2m}{\hbar^2} \frac{1}{2} m \omega^2 \text{ or } \alpha = \frac{m \omega}{2\hbar} \]

& the other match gives \[ 2\alpha = \frac{2m}{\hbar^2} E \], substituting \( \alpha \Rightarrow \]

\[ E = \frac{1}{2} \hbar \omega = hf \] !!!!!.....(Planck's Oscillators)

What about \( C_0 \)? We learn about that from the Normalization cond.

SHO: Normalization Condition

\[ \int_{-\infty}^{\infty} |\psi(x)|^2 \, dx = 1 = \int_{-\infty}^{\infty} C_0^2 e^{-\alpha x^2} \, dx \]

Since \[ \int_{-\infty}^{\infty} e^{-\alpha x^2} \, dx = \sqrt{\frac{\pi}{\alpha}} \] (don't memorize this)

Identifying \( \alpha = \frac{\hbar}{m} \) and using the identity above

\[ \Rightarrow C_0 = \sqrt{\frac{\pi}{\alpha}} \]

Hence the Complete NORMALIZED wave function is:

\[ \psi(x) = \sqrt{\frac{\pi}{\alpha}} e^{-\alpha x^2} \]

Ground State Wavefunction

has energy \( E = hf \)

Planck's Oscillators were solutions tied by the "spring" of the mutually attractive Coulomb Force.
Quantum Oscillator In Pictures

\[ E = KE + U(x) > 0 \text{ for } n=0 \]

Classically particle most likely to be at the turning point (velocity=0)
Quantum Mechanically, particle most likely to be at \( x=x_0 \) for \( n=0 \)

Classical & Quantum Pictures of SHO compared

- **Limits of classical vibration**: Turning Points (do on Board)
- **Quantum Probability** for particle outside classical turning points \( P(|x|>A) = 16\% \)
  - Do it on the board (see Example problems in book)
Excited States of The Quantum Oscillator

\[ \psi_n(x) = C_n H_n(x) e^{-\frac{x^2}{2\hbar}}; \]

\[ H_n(x) = \text{Hermite Polynomials with} \]
\[ H_0(x) = 1 \]
\[ H_1(x) = 2x \]
\[ H_2(x) = 4x^2 - 2 \]
\[ H_3(x) = 8x^3 - 12x \]
\[ H_n(x) = (-1)^n e^{-x^2} \frac{d^n e^{-x^2}}{dx^n} \]

and
\[ E_n = (n + \frac{1}{2})\hbar\omega = (n + \frac{1}{2})\hbar\omega \]

Again \( n = 0, 1, 2, 3 \ldots \infty \) Quantum #

Excited States of The Quantum Oscillator

Ground State Energy \( \geq 0 \) always