Physics 2D, Winter 2005

Week 9 Exercise Solutions

6-35 Applying the momentum operator $[p_x] = \left(\frac{\hbar}{i}\right)\frac{d}{dx}$ to each of the candidate functions yields

(a) $[p_x]\{A\sin(kx)\} = \left(\frac{\hbar}{i}\right)k\{A\cos(kx)\}$

(b) $[p_x]\{A\sin(kx) - A\cos(kx)\} = \left(\frac{\hbar}{i}\right)k\{A\cos(kx) + A\sin(kx)\}$

(c) $[p_x]\{A\cos(kx) + iA\sin(kx)\} = \left(\frac{\hbar}{i}\right)k\{-A\sin(kx) + iA\cos(kx)\}$

(d) $[p_x]\{e^{ik(x-a)}\} = \left(\frac{\hbar}{i}\right)ik\{e^{ik(x-a)}\}$

In case (c), the result is a multiple of the original function, since

$$-A\sin(kx) + iA\cos(kx) = i\{A\cos(kx) + iA\sin(kx)\}.$$

The multiple is $\left(\frac{\hbar}{i}\right)(ik) = \hbar k$ and is the eigenvalue. Likewise for (d), the operation $[p_x]$ returns the original function with the multiplier $\hbar k$. Thus, (c) and (d) are eigenfunctions of $[p_x]$ with eigenvalue $\hbar k$, whereas (a) and (b) are not eigenfunctions of this operator.

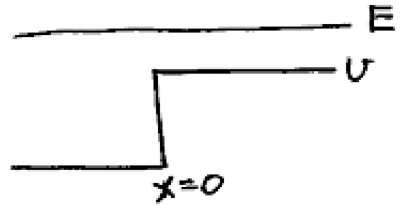
7-1 (a) The reflection coefficient is the ratio of the reflected intensity to the incident wave intensity, or $R = \frac{\left|(1/2)(1-i)\right|^2}{\left|(1/2)(1+i)\right|^2}$. But $|1-i|^2 = (1-i)(1-i)^* = (1-i)(1+i) = |1+i|^2 = 2$, so that $R = 1$ in this case.

(b) To the left of the step the particle is free. The solutions to Schrödinger's equation are $e^{\pm ikx}$ with wavenumber $k = \left(\frac{2mE}{\hbar^2}\right)^{1/2}$. To the right of the step $U(x) = U$ and the equation is $\frac{d^2\psi}{dx^2} = \frac{2m}{\hbar^2}(U-E)\psi(x)$. With $\psi(x) = e^{-kx}$, we find $\frac{d^2\psi}{dx^2} = k^2\psi(x)$, so that $k = \left[\frac{2m(U-E)}{\hbar^2}\right]^{1/2}$. Substituting $k = \left(\frac{2mE}{\hbar^2}\right)^{1/2}$ shows that $\left[\frac{E}{(U-E)}\right]^{1/2} = 1$ or $\frac{E}{U} = \frac{1}{2}$.

(c) For 10 MeV protons, $E = 10$ MeV and $m = \frac{938.28\text{ MeV}}{c^2}$. Using $\hbar = 197.3\text{ MeV fm}/c\left(1\text{ fm} = 10^{-15}\text{ m}\right)$, we find

$$\delta = \frac{1}{k} = \frac{\hbar}{(2mE)^{1/2}} = \frac{197.3\text{ MeV fm}/c}{\left[(2)\left(938.28\text{ MeV}/c^2\right)(10\text{ MeV})\right]^{1/2}} = 1.44\text{ fm}.$$

2



Sch. eqn: $-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V(x)\psi = E\psi.$

To left, V=0: $\frac{\partial^2\psi}{\partial x^2} = -k_1^2\psi$, $k_1^2 = \frac{2mE}{\hbar^2}$

To right, V=0: $\frac{\partial^2\psi}{\partial x^2} = -k_2^2\psi$, $k_2^2 = \frac{2m(E-U)}{\hbar^2}$.

Say have incoming wave from left:

$$\psi(x) = \begin{cases} Ae^{ik_1x} + Be^{-ik_1x} & x<0 \\ Ce^{ik_2x} & x>0 \end{cases}$$

(no e^{-ik_2x} term b/c nothing else to reflect off of).

Continuity: ψ must be single-valued at x=0, so

$A+B=C.$

Smoothness: Derivs must match, so $k_1(A-B)=k_2C$

$\Rightarrow C=\frac{2k_1}{k_1+k_2}A$, $B=\frac{k_1-k_2}{k_1+k_2}A$

So $\boxed{\psi(x)=\begin{cases} Ae^{ik_1x}+\frac{k_1-k_2}{k_1+k_2}Ae^{-ik_1x} & x>0 \\ A\frac{2k_1}{k_1+k_2}e^{ik_2x} & x<0 \end{cases}}$

b) $R=\frac{|B|^2}{|A|^2}=\boxed{\frac{(k_1-k_2)^2}{(k_1+k_2)^2}}$

$T=1-R$, so $T=1-\frac{(k_1-k_2)^2}{(k_1+k_2)^2}=\boxed{\frac{4k_1k_2}{(k_1+k_2)^2}}$

c) $E\to U$: $k_1\to\frac{2mU}{\hbar^2}$, $k_2\to 0$.

So $\boxed{T\to 0,\ R\to 1}$

$E\to\infty$: $k_1\to k_2$, so $\boxed{T\to 1,\ R\to 0}$

Makes sense: As E gets lower than U, no transmitted wave, and as $E\to\infty$, the step is negligible.

7-3

$$R = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2} = \frac{\left(1 - \frac{k_2}{k_1}\right)^2}{\left(1 + \frac{k_2}{k_1}\right)^2}$$

$$\frac{k_2}{k_1} = \sqrt{\frac{E-U}{E}} = \sqrt{5}, \text{ so } R = \frac{(1-\sqrt{5})^2}{(1+\sqrt{5})^2} = \boxed{0.146}$$

$$T = \frac{4\frac{k_2}{k_1}}{\left(1 + \frac{k_2}{k_1}\right)^2} = \frac{4\sqrt{5}}{(1+\sqrt{5})^2} = \boxed{0.854} \text{ (or } T = 1 - R\text{)}.$$

This is unchanged for electrons of the same energy.

7-5

We know

$$T(E) = \frac{1}{1 + \frac{1}{4}\frac{U^2}{E(U-E)}\sinh^2 \alpha L}, \quad \text{with } \alpha = \frac{\sqrt{2m(U-E)}}{\hbar}$$

Now, $\alpha L >> 1$, so $\sinh^2 \alpha L = \left(\frac{e^{\alpha L} - e^{-\alpha L}}{2}\right)^2 \simeq \frac{1}{4} e^{2\alpha L}$.

Also, $\frac{U^2}{E(U-E)} = \frac{U}{E(1-\frac{E}{U})} \approx \frac{U}{E}$. $>> 1$. so

$$T(E) \simeq \frac{1}{1 + \frac{1}{16}\frac{U}{E} e^{2\alpha L}} \simeq \boxed{16 \frac{E}{U} e^{-2\alpha L}}.$$

b] exp. factor is $e^{-2\alpha L} = e^{-2\left[\sqrt{2m(U-E)}/\hbar\right] L}$

1) $\alpha = \sqrt{\frac{2m_e(0.01\,eV)}{(\hbar c)^2}} = .51\ nm^{-1}$ ($m_e = .511\ MeV$)

$$\boxed{e^{-2\alpha L} = 0.90}$$

2) $\alpha = \sqrt{\frac{2m_e(1\,eV)}{(\hbar c)^2}} = 5.1\ nm^{-1}$

$$\boxed{e^{-2\alpha L} = 0.36}$$

3) $\alpha = \sqrt{\frac{2m_e(10^6\,eV)}{\hbar^2}} = 4.4 \times 10^{14}\ m^{-1}$

$$\boxed{e^{-2\alpha L} = 0.41}$$

4) $\alpha = \sqrt{\frac{2m(1\,J)}{\hbar}} = 3.8 \times 10^{34}\ m^{-1}$, $\boxed{e^{-2\alpha L} \simeq 0}$ (Classically, ball can't hop barrier).

7-7 The continuity requirements from Equation 7.8 are

$$[\text{continuity of } \Psi \text{ at } x=0] \qquad A+B=C+D$$

$$\left[\text{continuity of } \frac{\partial\Psi}{\partial x} \text{ at } x=0\right] \qquad ikA-ikB=\alpha D-\alpha C$$

$$[\text{continuity of } \Psi \text{ at } x=L] \qquad Ce^{-\alpha L}+De^{+\alpha L}=Fe^{ikL}$$

$$\left[\text{continuity of } \frac{\partial\Psi}{\partial x} \text{ at } x=L\right] \qquad \alpha De^{+\alpha L}-\alpha Ce^{-\alpha L}=ikFe^{ikL}$$

To isolate the transmission amplitude $\frac{F}{A}$, we must eliminate from these relations the unwanted coefficients *B, C,* and *D*. Dividing the second line by *ik* and adding to the first eliminates *B*, leaving *A* in terms of *C* and *D*. In the same way, dividing the fourth line by α and adding the result to the third line gives *D* (in terms of *F*), while subtracting the result from the third line gives *C* (in terms of *F*). Combining these results finally yields *A*: $A=\frac{1}{4}Fe^{ikL}\left\{\left[2-\left(\frac{\alpha}{ik}+\frac{ik}{\alpha}\right)\right]e^{+\alpha L}+\left[2+\left(\frac{\alpha}{ik}+\frac{ik}{\alpha}\right)\right]e^{-\alpha L}\right\}$. The transmission probability is $T=\left|\frac{F}{A}\right|^2$. Making use of the identities $e^{\pm\alpha L}=\cosh\alpha L\pm\sinh\alpha L$ and $\cosh^2\alpha L=1+\sinh^2\alpha L$, we obtain

$$\frac{1}{T}=\left|\frac{A}{F}\right|^2=\frac{1}{4}\left|2\cosh\alpha L+i\left(\frac{\alpha}{k}-\frac{k}{\alpha}\right)\sinh\alpha L\right|^2=\cosh^2\alpha L+\frac{1}{4}\left(\frac{\alpha}{k}-\frac{k}{\alpha}\right)^2\sinh\alpha L$$

$$=1+\frac{1}{4}\left[\frac{U-E}{E}+\frac{E}{U-E}+2\right]\sinh^2\alpha L=1+\frac{1}{4}\left[\frac{U^2}{E(U-E)}\right]\sinh^2\alpha L$$