# Physics 2D, Winter 2005

## Week 8 Exercise Solutions

6-14

We know that $E = \sqrt{p^2c^2 + m^2c^4}$

So using $p = \frac{n\hbar\pi}{L}$, we get

$$\boxed{E = \sqrt{\frac{n^2\hbar^2\pi^2c^2}{L^2} + m^2c^4}}$$

Ground state: $n=1$, so $E = \sqrt{\frac{\hbar^2\pi^2c^2}{L^2} + m^2c^4}$

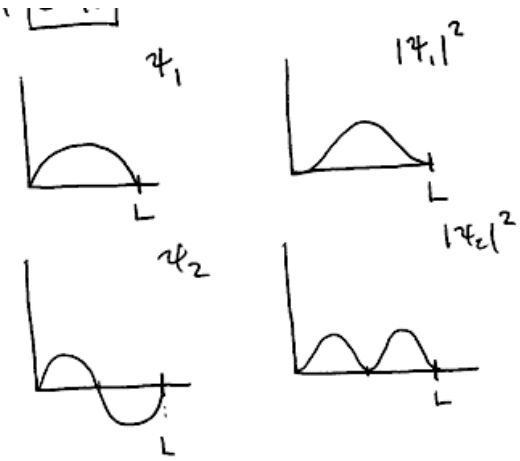
Using $L = 10^{-12}$ m, $m = m_e$, we get $E = 804$ keV

Thus $KE = E - mc^2 = \boxed{293 \text{ keV}}$

The non-rel. answer is $KE = \frac{\pi^2\hbar^2}{2mL^2} = 377$ keV, off by almost $\boxed{30\%}$!

6-17

a)



b) Prob of finding electron bet. $x=a$ and $x=b$ is

$$\int_a^b |\psi(x)|^2 dx = \int_a^b |\psi_1(x)|^2 dx$$

$$\psi_1(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{\pi x}{L}\right), \text{ so}$$

$$\text{Prob} = \frac{2}{L}\int_a^b \sin^2\left(\frac{\pi x}{L}\right) dx$$

Can do the integral by using $u = \frac{\pi x}{L}$

$$\rightarrow \frac{2}{L}\int_{\frac{\pi a}{L}}^{\frac{\pi b}{L}} \sin^2 u \cdot \frac{L}{\pi} du = \frac{2}{\pi}\int_{\frac{\pi a}{L}}^{\frac{\pi b}{L}} \sin^2 u \, du$$

$$= \frac{2}{\pi}\left[\frac{u}{2} - \frac{\sin 2u}{4}\right]_{\pi a/L}^{\pi b/L}$$

$$= \frac{2}{\pi}\cdot\frac{1}{2}\left(\frac{\pi b}{L} - \frac{\pi a}{L}\right) + \frac{2}{\pi}\frac{1}{4}\left(\sin\left(\frac{2\pi a}{L}\right) - \sin\left(\frac{2\pi b}{L}\right)\right)$$

Plugging in $L = 1\text{nm}$, $a = .15\text{nm}$, $b = .35\text{nm}$ gives

$$\text{Prob} = (.35 - .15) + \frac{1}{2\pi}\left(\sin(.3\pi) - \sin(.7\pi)\right) = \boxed{\frac{2}{10}}$$

c] Now Prob $= \int_a^b |\psi_2(x)|^2 dx = \frac{2}{L}\int_a^b \sin^2\left(\frac{2\pi x}{L}\right) dx$

$$= \frac{1}{\pi}\int_{\frac{2\pi a}{L}}^{\frac{2\pi b}{L}} \sin^2 u \, du$$

$$= \frac{1}{2\pi}\left(\frac{2\pi b}{L} - \frac{2\pi a}{L}\right) + \frac{1}{\pi 4}\left(\sin\left(\frac{4\pi a}{L}\right) - \sin\left(\frac{4\pi b}{L}\right)\right)$$

$$= \left(\frac{b}{L} - \frac{a}{L}\right) + \frac{1}{4\pi}\left(\sin\left(\frac{4\pi a}{L}\right) - \sin\left(\frac{4\pi b}{L}\right)\right)$$

which here $= \boxed{0.35}$

d] $E_n = \frac{n^2\pi^2\hbar^2}{2mL^2} = \frac{n^2\pi^2(\hbar c)^2}{2mc^2L^2}$.

Using $(\hbar c) = 197.3\,\text{eV}\cdot\text{nm}$, we get

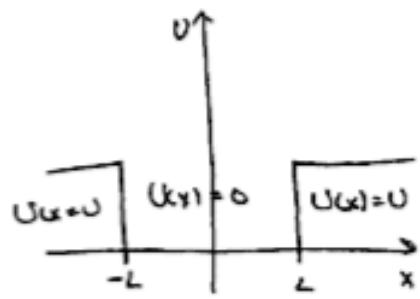
$\boxed{E_1 = 0.376\,\text{eV}}$, $\boxed{E_2 = 4E_1 = 1.5\,\text{eV}}$

6-20

Consider a particle with energy E bound to a finite square well of height U and width 2L situated on $-L \le x \le L$. Because the potential energy is symmetric about the midpoint of the well, the stationary state waves will be either symmetric (cosine solutions) or antisymmetric (sine solutions) about this point.

a.) Show that for $E < U$, the conditions for smooth joining of the interior and exterior waves lead to the following equation for the allowed energies of the symmetric waves.

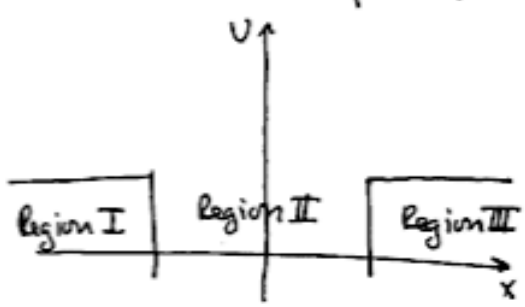
Let me start out by solving the finite square well problem for ya. Our box now looks like:




Since the potential is finite, there is some probability to find the particle in those regions. To find out how much, whip out the Schrödinger equation:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + U(x)\psi(x) = E\psi(x)$$

We need to divide up our problem into three regions:



In Region I, the potential $U(x)$ is $U(x) = U$, so the Schrödinger Eq. is:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + U\psi = E\psi \qquad \underline{\text{Region I}}$$

In Region II, the potential is zero, so we get

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi \qquad \underline{\text{Region II}}$$

Region III is the same as I:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + U\psi = E\psi \qquad \text{Region III}$$

The solution to region II is easiest, cuz that's what you guys did last week:

$$\text{Region II}: \quad \frac{d^2\psi}{dx^2} + \frac{2mE}{\hbar^2}\psi = 0 = \frac{d^2\psi}{dx^2} + k^2\psi \qquad k^2 = \frac{2mE}{\hbar^2}$$

$$\Rightarrow \psi = A\sin(kx) + B\cos(kx)$$

The solutions to Regions I and III are just a little more complicated

$$\text{Region I III}: \quad -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + U\psi = E\psi$$

$$\frac{d^2\psi}{dx^2} + \frac{2m(E-U)}{\hbar^2}\psi = 0$$

$$\frac{d^2\psi}{dx^2} + k'^2\psi = 0 \qquad k'^2 = \frac{2m(E-U)}{\hbar^2}$$

So far, it's exactly the same as Region II, except with the U thrown in. However, there's a little complication. We've told $E < U$, so that means $k''^2$ is negative. We can make it explicit by saying

$$k''^2 = \frac{2m(E-U)}{\hbar^2} = -\frac{2m(U-E)}{\hbar^2}$$

That means $k'$ has to be imaginary!

$$k' = i\sqrt{\frac{2m(U-E)}{\hbar^2}} = i\alpha \quad ; \quad \alpha^2 = \frac{2m(U-E)}{\hbar^2}$$

So we have

$$\frac{d^2\psi}{dx^2} + k''^2\psi = 0$$

$$\Rightarrow \psi = Ce^{ik'x} + De^{-ik'x}$$

$$= Ce^{-\alpha x} + De^{\alpha x}$$ (could also be $C\sinh(\alpha x) + D\cosh(\alpha x)$ if you're into hyperbolic functions)

So let's list our solutions:

Region I: $\psi(x) = Ae^{\alpha x} + Be^{-\alpha x}$ $\qquad \alpha^2 = \frac{2m(U-E)}{\hbar^2}$

Region II: $\psi(x) = C\sin(kx) + D\cos(kx)$ $\qquad k^2 = \frac{2mE}{\hbar^2}$

Region III: $\psi(x) = Ee^{\alpha x} + Fe^{-\alpha x}$

We can simplify some stuff by demand $\psi$ be finite.

In region I, as $x \to \infty$, we want $\psi$ to be finite (go to zero)

So

Region I: $\psi(x \to -\infty) = \lim_{x \to -\infty} \left( Ae^{\alpha x} + Be^{-\alpha x} \right) = 0$

$$\lim_{x \to -\infty} e^{\alpha x} = 0$$

$$\lim_{x \to -\infty} e^{-\alpha x} = \infty$$

So to keep $\psi$ finite,

$$B = 0$$

$\therefore$ In region I,

$$\psi(x) = Ae^{\alpha x}$$

Similarly, to keep region III finite as $x \to +\infty$,

Region III: $\psi(x \to \infty) = \lim_{x \to +\infty} \left( Ee^{\alpha x} + Fe^{-\alpha x} \right) = 0$

$$\lim_{x \to +\infty} e^{\alpha x} = \infty$$

$$\lim_{x \to +\infty} e^{-\alpha x} = 0$$

$$\therefore E = 0$$

So in Region III,

$$\psi(x) = Fe^{-\alpha x}$$

Plus, in this problem, we're only worried about the symmetric (cosine) states. So now we're left with

Region I : $\psi(x) = Ae^{\alpha x}$

Region II : $\psi(x) = D\cos(kx)$

Region III : $\psi(x) = Fe^{-\alpha x}$

At the boundaries, we want to force $\psi$ to be continuous, i.e.,

<u>at $x = -L$</u>

$$Ae^{-\alpha L} = D\cos(kL) \qquad (1)$$

<u>at $x = +L$</u>

$$Fe^{-\alpha L} = D\cos(kL) \qquad (2)$$

We also want $\psi$ to be <u>smooth</u> at the boundries (smooth = $\frac{d\psi}{dx}$ is continuous)

<u>at $x = -L$</u>

$$\frac{d\psi}{dx}_{\text{Region I}} = \frac{d\psi}{dx}_{\text{Region II}}$$

$$\Rightarrow A\alpha e^{-\alpha L} = -Dk\sin(-kL) \qquad (3)$$

$$= Dk\sin(kL)$$

at $x=L$

$$\frac{d\psi}{dx}\Big|_{\text{Region III}} = \frac{d\psi}{dx}\Big|_{\text{Region II}}$$

$$\Longrightarrow -F\alpha e^{-\alpha L} = -D\sin(kL) \qquad (4)$$

Divide eq. (3) by (1) and (4) by (2) to get

$$\alpha = k\frac{\sin(kL)}{\cos(kL)}$$

$$\therefore \boxed{k\tan(kL) = \alpha} \quad \blacksquare$$

b) Show that the energy condition in a.) can be written as

$$k\sec(kL) = \sqrt{\frac{2mU}{\hbar^2}}$$

Since $\alpha^2 = \frac{2mU}{\hbar^2} - \frac{2mE}{\hbar^2}$

$$k^2 = \frac{2mE}{\hbar^2}$$

$$\alpha^2 + k^2 = \frac{2mU}{\hbar^2}$$

$$\therefore \alpha = \sqrt{\frac{2mU}{\hbar^2} - k^2}$$

$$\therefore \; k \tan(kL) = \sqrt{\frac{2mU}{\hbar^2} - k^2}$$

$$k^2 \tan^2(kL) = \frac{2mU}{\hbar^2} - k^2$$

$$k^2\left(\tan^2(kL) + 1\right) = \frac{2mU}{\hbar^2}$$

Use the trig identity

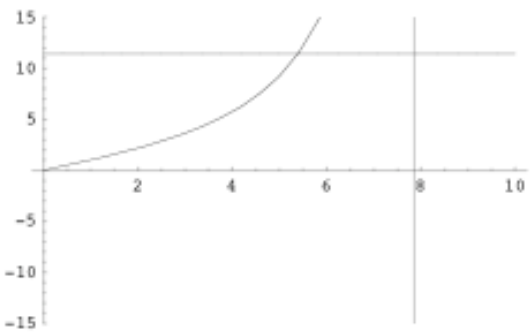
$$\tan^2\theta + 1 = \sec^2\theta$$

$$\therefore \; \boxed{k \sec(kL) = \sqrt{\frac{2mU}{\hbar^2}}} \; \blacksquare$$

The easiest way to solve something an equation numerically is to plot both sides and find the intersection. The right side of the equation is just

$$\frac{\sqrt{2\,m\,U}}{\hbar} = \frac{\sqrt{2\,m\,c^2\,U}}{\frac{1}{2\pi}\,h\,c} = 2\,\pi\,\frac{\sqrt{2\,(0.511\text{ MeV})\,(5\text{ eV})}}{1240\text{ eV nm}} \approx 11.454\,\frac{1}{\text{nm}}$$

Here is my plot of $y = k \sec k\,L$ for $L = 0.2$ nm and $y = 11.45\,\frac{1}{\text{nm}}$.
The horizontal axis is $k$ and the vertical axis is the two sides of the equation, both in units of $\text{nm}^{-1}$.




As you can see if you look closely, the two curves intersect for $k \approx 5.39958\text{ nm}^{-1}$. Let's make sure this number is about right by plugging it in.

$$(5.39958)\sec(\,(5.39958)\,(0.2)\,) \approx 11.4543$$

Bingo! Now, to get $E$ we just solve $k = \frac{\sqrt{2\,m\,E}}{\hbar}$.

$$E = \hbar^2\,k^2\,/\,2\,m$$

That should look familiar. Plugging in our value of $k$ gives

$$E = \frac{(h\,c)^2\,k^2}{(2\,\pi)^2\,(m\,c^2)} = \frac{(1240\text{ eV nm})^2\,(5.39958\text{ nm}^{-1})^2}{(2\,\pi)^2\,(511000\text{ eV})} \approx 2.222\text{ eV}$$

6-21


6-24

a)

Schrödinger's equation for a quantum oscillator is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2}m\omega^2 x^2 \psi = E\psi \qquad \text{(eq. 5.25)}$$

Take derivatives:

$$\frac{d^2\psi}{dx^2} : \frac{d^2}{dx^2}\left(Cxe^{-\alpha x^2}\right) = \frac{d}{dx}\left(Ce^{-\alpha x^2} - 2Cx^2\alpha e^{-\alpha x^2}\right)$$

$$= -2C\alpha x e^{-\alpha x^2} - 4Cx\alpha e^{-\alpha x^2} + 4Cx^3\alpha^2 e^{-\alpha x^2}$$

$$= 4C\alpha^2 x^3 e^{-\alpha x^2} - 6C\alpha x e^{-\alpha x^2}$$

So Erwin's equation up there looks like

$$-\frac{\hbar^2}{2m}\left(4C\alpha^2 x^3 - 6C\alpha x\right)e^{-\alpha x^2} + \frac{1}{2}m\omega^2\left(Cx^3 e^{-\alpha x^2}\right)$$

$$= ECxe^{-\alpha x^2}$$

All those terms have C's and $e^{-\alpha x^2}$'s in 'em so cancel those out and rearrange to get

$$4\alpha^2 x^2 - 6\alpha = \left(\frac{m\omega}{\hbar}\right)^2 x^2 - \frac{2mE}{\hbar^2}$$

Equate coefficients of like powers of x:

$$4\alpha^2 = \left(\frac{m\omega}{\hbar}\right)^2$$

$$\Rightarrow \boxed{\alpha = \frac{m\omega}{2\hbar}}$$

and

$$6\alpha = \frac{2mE}{\hbar^2}$$

$$\therefore \boxed{E = \frac{3\alpha\hbar^2}{m} = \frac{3}{2}\hbar\omega}$$

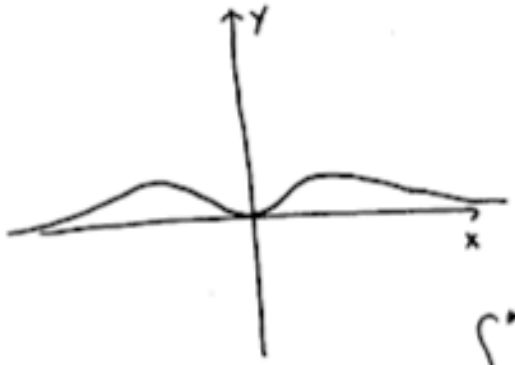
b.) Normalize this wave

Okay dokey. The thing can be anywhere between $-\infty < x < \infty$, So

$$P = \int_{-\infty}^{\infty} |\psi(x)|^2 dx = 1$$

$$= C^2 \int_{-\infty}^{\infty} x^2 e^{-2\alpha x^2} dx$$

To do this, we can use a keen little math trick. Note that the integrand is symmetric under $x \to -x$. If you plot it, it looks like:




See? It's symmetric about the y-axis. Now, an integral gives you the area under the curve, but since the area from $-\infty$ to zero is the same as from zero to $\infty$, we can write

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx$$
$$= \int_{0}^{\infty} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx$$

$$\therefore \int_{-\infty}^{\infty} f(x)\,dx = 2\int_{0}^{\infty} f(x)\,dx \quad \text{if} \quad f(x) = f(-x)$$

Sneaky! So now we can say

$$P = 2C^2 \int_0^{\infty} x^2 e^{-2\alpha x^2}\,dx$$

Still pretty hairy. The easiest thing to do is look it up, but you masochists out there can try to integrate by parts. Pain hurts me, though, so I'll look it up and say

$$P = 2C^2 \int_0^{\infty} x^2 e^{-2\alpha x^2}\,dx = 2C^2 \left( \frac{1}{8\alpha}\sqrt{\frac{\pi}{2\alpha}} \right) = 1$$

So,

$$\boxed{C = \left(\frac{32\alpha^3}{\pi}\right)^{1/4}}$$

6-29

a.) Find the value of C that normalizes $\psi$

$$P = \int_0^\infty C^2 e^{-2x}(1-e^{-x})^2 dx$$

$$= C^2 \int_0^\infty (e^{-2x} - 2e^{-3x} + e^{-4x}) dx$$

$$= C^2 \left[ -\frac{e^{-2x}}{2} + \frac{2e^{-3x}}{3} - \frac{e^{-4x}}{4} \right] \Bigg|_0^\infty$$

$$= C^2 \left[ \frac{1}{2} - \frac{2}{3} + \frac{1}{4} \right] = \frac{C^2}{12} = 1$$

$$\therefore \boxed{C = \sqrt{12}}$$

b.) Where is the electron most likely to be found? That is, for what value of x is the probability of finding the electron largest?

The probability density is

$$|\psi(x)|^2 = 12(e^{-2x} - 2e^{-3x} + e^{-4x})$$

Maximize!

$$\frac{d|\psi(x)|^2}{dx} = 12(-2e^{-2x} + 6e^{-3x} - 4e^{-4x}) = 0$$

$$\Rightarrow -1 + 3e^{-x} - 2e^{-2x} = 0$$

let $u = e^{-x}$

$$= 2u^2 - 3u + 1 = 0$$

$$\Rightarrow u = \frac{3 \pm \sqrt{9-8}}{4} = \frac{3 \pm 1}{4} = 1, \frac{1}{2}$$

$u = e^{-x} = 1$

$\Rightarrow x = 0$ min

$u = e^{-x} = \frac{1}{2}$

$\Rightarrow \boxed{x = \ln 2 \quad \text{max}}$

c.) calculate $\langle x \rangle$ for this electron and compare your results with its most likely position. Comment on any differences you find.

$$\langle x \rangle = \int_0^\infty \psi^* x \psi dx = C^2 \int_0^\infty x(e^{-2x} - 2e^{-3x} + e^{-4x}) dx$$

We need to know how to do an integral of the form

$$\int x e^{-ax} dx$$

When in doubt, there's no reason to be subtle. . Integrate it by parts:

$$\text{let } u = x \qquad dv = e^{-ax} dx$$
$$dx = du \qquad v = -\frac{e^{-ax}}{a}$$

$$\Rightarrow \int x e^{-ax} dx = -\frac{x e^{-ax}}{a} + \frac{1}{a}\int e^{-ax} dx$$

$$= -\frac{x e^{-ax}}{a} - \frac{e^{-ax}}{a^2}$$

$$\therefore \int_0^\infty x e^{-ax} dx = -\frac{x e^{-ax}}{a}\Big|_0^\infty - \frac{e^{-ax}}{a^2}\Big|_0^\infty$$

$$= \frac{1}{a^2}$$

So,

$$\langle x \rangle = C^2\left(\frac{1}{4} - \frac{2}{9} + \frac{1}{16}\right) = \frac{13}{12}$$

$$\boxed{\langle x \rangle = 1.083 \text{ nm}}$$

6-30

The possible particle positions within the box are weighted according to the probability density $|\psi|^2 = \frac{2}{L}\sin^2\left(\frac{n\pi x}{L}\right)$. The position is calculated as

$$\langle x\rangle = \int_0^L x|\psi|^2\,dx = \frac{2}{L}\int_0^L x\sin^2\left(\frac{n\pi x}{L}\right)dx\,.$$

Making the change of variable $\theta = \frac{nx}{L}$ (so that $d\theta = \frac{\pi\,dx}{L}$) gives

$$\langle x\rangle = \frac{2L}{\pi^2}\int_0^\pi \theta\sin^2 n\theta\,d\theta\,.$$

Using the trigonometric identity $2\sin^2\theta = 1-\cos 2\theta$, we get

$$\langle x\rangle = \frac{L}{\pi^2}\left\{\int_0^\pi \theta\,d\theta - \int_0^\pi \theta\cos 2n\theta\,d\theta\right\}.$$

An integration by parts shows that the second integral vanishes, while the first integrates to $\frac{\pi^2}{2}$. Thus, $\langle x\rangle = \frac{L}{2}$, independent of $n$.

For the computation of $\langle x^2\rangle$, there is an extra factor of $x$ in the integrand. After changing variables to $\theta = \frac{\pi x}{L}$ we get $\langle x^2\rangle = \frac{L^2}{\pi^3}\left\{\int_0^\pi \theta^2\,d\theta - \int_0^\pi \theta^2\cos 2n\theta\,d\theta\right\}$. The first integral evaluates to $\frac{\pi^3}{3}$, the second may be integrated by parts twice to get

$$\int_0^\pi \theta^2\cos 2n\theta\,d\theta = -\frac{1}{n}\int_0^\pi \theta\sin 2n\theta\,d\theta = \left(\frac{1}{2n^2}\right)\theta\cos 2n\theta\Big|_0^\pi = \frac{\pi}{2n^2}\,.$$

Then $\langle x^2\rangle = \frac{L^2}{\pi^3}\left\{\frac{\pi^3}{3} - \frac{\pi}{2n^2}\right\} = \frac{L^2}{3} - \frac{L^2}{2(n\pi)^2}$.

6-32

The ground state wavefunction of the quantum oscillator is:

$$\psi(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left[-\frac{m\omega}{2\hbar}x^2\right]$$

So

$$\langle x\rangle = \int_{-\infty}^{\infty}\psi^* x\psi\,dx = \left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\int_{-\infty}^{\infty} x\,e^{-\frac{m\omega}{\hbar}x^2}\,dx$$

Again, the integrand is odd, and the interval is symmetric

$$\therefore \boxed{\langle x \rangle = 0}$$

$$\langle x^2 \rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \int_{-\infty}^{\infty} x^2 e^{-\frac{m\omega}{\hbar}x^2} dx$$

Now the integrand is even, so

$$= 2\left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \int_{0}^{\infty} x^2 e^{-\frac{m\omega}{\hbar}x^2} dx$$

Now let's use their hint:

$$\int_0^{\infty} x^2 e^{-ax^2} dx = \frac{1}{4a}\sqrt{\frac{\pi}{a}}$$

with $a = \frac{m\omega}{\hbar}$

$$\boxed{\langle x^2 \rangle = \frac{\hbar}{2m\omega}}$$

$$\boxed{\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \left(\frac{\hbar}{2m\omega}\right)^{1/2}}$$

6-33

a) We expect $\langle p_x \rangle = 0$, since the particle is oscillating – just moves back and forth.

b) We know $E = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$.

So $\langle E \rangle = \frac{\langle p^2 \rangle}{2m} + \frac{1}{2}m\omega^2 \langle x^2 \rangle$

Since $\langle E \rangle = \frac{1}{2}\hbar\omega$ for the ground state,

$$\frac{1}{2}\hbar\omega = \frac{\langle p^2 \rangle}{2m} + \frac{1}{2}m\omega^2 \langle x^2 \rangle$$

$$\Rightarrow \langle p^2 \rangle = 2m\left[\frac{1}{2}\hbar\omega - \frac{1}{2}m\omega^2\langle x^2 \rangle\right]$$

$$= m\hbar\omega - m^2\omega^2\langle x^2 \rangle.$$

We know $\langle x^2 \rangle = \frac{\hbar}{2m\omega}$, so $\boxed{\langle p^2 \rangle = \frac{\hbar m\omega}{2}}$

c) $\Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \boxed{\left(\frac{\hbar m\omega}{2}\right)^{1/2}}$

Notice: $\Delta p \, \Delta x = \left(\frac{\hbar m\omega}{2}\right)^{1/2}\left(\frac{\hbar}{2m\omega}\right)^{1/2} = \frac{\hbar}{2}$ (The minimum uncertainty!)

6-37

a) Show that the value $C = \frac{1}{\sqrt{2}}$ normalizes this wave, assuming $\psi_1$ and $\psi_2$ are themselves normalized

First of all, you need to know that the $\psi_n(x)$'s of the square well are what's called <u>orthogonal</u>. That is,

$$\int_0^L \psi_n^*(x)\,\psi_m(x)\,dx = \begin{cases} 0 & \text{if } n \neq m \\ 1 & \text{if } n = m \end{cases}$$

You can show this using

$$\psi(x) = \sqrt{\frac{2}{L}}\,\sin\left(\frac{n\pi x}{L}\right)$$

But I'm not going to. Take my word for it. You might want to try proving it, though.

So let's normalize $\Psi(x,0)$

$$P = \int_0^L \Psi^* \Psi\,dx = \int_0^L C^2\left[\psi_1^*(x) + \psi_2^*(x)\right]\left[\psi_1(x) + \psi_2(x)\right]dx$$

$$= C^2 \int_0^L [\psi_1^* \psi_1 + \psi_2^* \psi_2 + \psi_1^* \psi_2 + \psi_2^* \psi_1] \, dx$$

Since the $\psi_n$'s are orthogonal, the last two terms vanish

$$= C^2 \int_0^L [\psi_1^* \psi_1 + \psi_2^* \psi_2] \, dx$$

$\psi_1$ and $\psi_2$ are themselves normalized, so this is

$$= C^2 (1+1) = 2C^2 = 1$$

$$\therefore \boxed{C = \frac{1}{\sqrt{2}}}$$

b.) Find $\Psi(x,t)$ at any later time $t$.

Remember that to get the time dependent solution, you just multiply by

$$\phi_n(t) = e^{-i \frac{E_n}{\hbar} t}$$

$$\Psi(x,t) = C [\psi_1(x,t) + \psi_2(x,t)]$$

$$\boxed{\Psi(x,t) = C \left[ \psi_1(x) e^{-i \frac{E_1}{\hbar} t} + \psi_2(x) e^{-i \frac{E_2}{\hbar} t} \right]}$$

c.) Show that the superposition is not a stationary state, but that the average energy in this state is the arithmetic mean $(E_1+E_2)/2$ of the ground and first excited state energies $E_1$ and $E_2$.

What is meant by a <u>stationary state</u> is that the probability density does not change in time. So the way to show this is if something <u>is</u> a stationary state, then with respect to time the probability density is constant. So

$$\frac{\partial|\psi(x,t)|^2}{\partial t}=0 \quad \text{if } \psi(x,t) \text{ is a stationary state}$$

Let's try it with this one

$$|\Psi(x,t)|^2 = C^2\left[\psi_1^*(x)\, e^{\frac{iE_1t}{\hbar}} + \psi_2^*\, e^{\frac{iE_2t}{\hbar}}\right]\left[\psi_1(x)\, e^{-\frac{iE_1t}{\hbar}} + \psi_2(x)\, e^{-\frac{iE_2t}{\hbar}}\right]$$

$$= C^2\Big[\psi_1^*(x)\psi_1(x) + \psi_2^*\psi_2(x) + \psi_1^*(x)\psi_2(x)\, e^{-\frac{i(E_2-E_1)t}{\hbar}} + \psi_2^*(x)\psi_1(x)\, e^{\frac{i(E_2-E_1)t}{\hbar}}\Big]$$

So

$$\boxed{\frac{\partial|\Psi(x,t)|^2}{\partial t} \neq 0\,!}$$

Not a stationary state!

To get the average energy

$$\langle E \rangle = \int_0^L \Psi^*(x,t)\, \hat{E}\, \Psi(x,t)\, dx$$

Remember that

$$\hat{E} = i\hbar \frac{\partial}{\partial t}$$

$$= C^2 \int_0^L \left[\psi_1^*(x) e^{\frac{iE_1 t}{\hbar}} + \psi_2^*(x) e^{\frac{iE_2 t}{\hbar}}\right]\left(i\hbar \frac{\partial}{\partial t}\right)\left[\psi_1(x) e^{-\frac{iE_1 t}{\hbar}} + \psi_2(x) e^{-\frac{iE_2 t}{\hbar}}\right] dx$$

$$= i\hbar C^2 \int_0^L \left[\psi_1^* e^{\frac{iE_1 t}{\hbar}} + \psi_2^* e^{\frac{iE_2 t}{\hbar}}\right]\left[-i\frac{E_1}{\hbar}\psi_1(x) e^{-\frac{iE_1 t}{\hbar}} - i\frac{E_2}{\hbar}\psi_2(x) e^{-\frac{iE_2 t}{\hbar}}\right] dx$$

$$= C^2 \int_0^L \left[E_1 \psi_1^* \psi_1 + E_2 \psi_2^* \psi_2\right] dx$$

All the other terms go away thanks to orthogonality!

$$= C^2 (E_1 + E_2)$$

$$= \boxed{\frac{(E_1 + E_2)}{2} = \langle E \rangle}$$

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$$\langle x \rangle = \int \Psi^*(x,t)\, x\, \Psi(x,t)$$

$$= \int \left(\frac{1}{\sqrt{2}}\psi_1^*(x) e^{+i\omega_1 t} + \psi_2^*(x) e^{+i\omega_2 t}\right) x \left(\frac{1}{\sqrt{2}}\right)\left(\psi_1(x) e^{-i\omega_1 t} + \psi_2(x) e^{-i\omega_2 t}\right)$$

$$= \frac{1}{2}\left(\int x|\psi_1|^2 dx + \int x|\psi_2|^2 dx\right) + \frac{1}{2}\int \left(\psi_1 \psi_2^* e^{i(\omega_2 - \omega_1)t} x + \psi_2 \psi_1^* e^{-i(\omega_1 - \omega_2)t} x\right) dx$$

Now, since $\psi_1$ & $\psi_2$ are real, $\psi_1\psi_2^* = \psi_2\psi_1^*$

<u>and</u> $e^{i\theta} + e^{-i\theta} = 2\cos\theta$, so

$$\frac{1}{2}\int\left(\psi_1\psi_2^* e^{i(\omega_2-\omega_1)t} + \psi_2\psi_1^* e^{-i(\omega_1-\omega_2)t}\right)dx = \frac{1}{2}\int x\,\psi_1^*\psi_2 \cdot 2\cos((\omega_2-\omega_1)t)\,dx$$

$$= \int x\,\psi_1^*\psi_2 \cos\left(\frac{E_2-E_1}{\hbar}t\right)$$

<u>Done!</u> $\langle x\rangle = \left[\frac{1}{2}\left(\int x|\psi_1|^2 dx + \int x|\psi_2|^2 dx\right)\right] + \left[\int x\,\psi_1^*\psi_2\,dx\right]\cos\left(\frac{E_2-E_1}{\hbar}t\right)$

For $L = 1\,nm$,

$$X_0 = \frac{1}{2}\left[\langle x\rangle_1 + \langle x\rangle_2\right] = \frac{1}{2}\left[\frac{L}{2} + \frac{L}{2}\right] = \boxed{\frac{L}{2}}$$

and $$A = \int x\,\psi_1^*\psi_2 = \frac{2}{L}\int_0^L x \sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{2\pi x}{L}\right)dx$$

$$= \frac{1}{L}\int_0^L x\left[\cos\left(\frac{\pi x}{L}\right) - \cos\left(\frac{3\pi x}{L}\right)\right]dx$$ (I used a trig identity).

Now integrate by parts:

$$\int x\cos(ax)\,dx = \frac{1}{a}x\sin(ax) - \int \frac{1}{a}\sin(ax)$$

$$\Rightarrow A = \frac{1}{L}\left[-\frac{L}{\pi}\int_0^L \sin\left(\frac{\pi x}{L}\right) + \frac{L}{3\pi}\int_0^L \sin\left(\frac{3\pi x}{L}\right)\right]$$

$$= +\left(\frac{L}{\pi}\right)^2(-2) - \left(\frac{L}{3\pi}\right)^2(-2)$$

$$= L^2\left(\frac{2}{9\pi^2} - \frac{2}{\pi^2}\right) = \boxed{\frac{-16L}{9\pi^2}} = -0.18\,nm \text{ for } L = 1\,nm.$$

Time to shuttle back 'n' forth = Period = $\frac{1}{F} = \frac{2\pi}{\omega}$

Here, $\omega = \frac{E_2 - E_1}{\hbar} = \frac{3E_1}{\hbar} = \frac{3\pi^2\hbar}{2mL^2}$

So $\frac{2\pi}{\omega} = \frac{4mL^2}{3\pi\hbar} = \boxed{3.67\times10^{-15}\,s}$

Classically, electron would have $v = \left(\frac{2E}{m}\right)^{1/2} = \left(\frac{E_1+E_2}{m}\right)^{1/2}$

$E_1 + E_2 = \frac{5\pi^2\hbar^2}{2mL^2}$, so $v = \left(\frac{5\pi^2\hbar^2}{2m^2L^2}\right)^{1/2} = \sqrt{\frac{5}{2}}\,\frac{\pi\hbar}{mL} = \boxed{5.75\times10^{5}\,\frac{m}{s}}$

So it would take time $2\left(\frac{L}{v}\right) = \boxed{3.47\times10^{-15}\,s}$ classically!