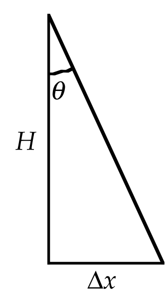
Physics 2D, Winter 2005

Week 7 Exercise Solutions

5-21 (a) The woman tries to hold a pellet within some horizontal region Δx_{i} and directly above the spot on the floor. The uncertainty principle requires her to give a pellet some x velocity at least as large as $\Delta v_x = \frac{\hbar}{2m\Delta x_{\mathrm{i}}}$. When the pellet hits the floor at time t, the total miss distance is $\Delta x_{\mathrm{total}} = \Delta x_{\mathrm{i}} + \Delta v_x t = \Delta x_{\mathrm{i}} + \left(\frac{\hbar}{2m\Delta x_{\mathrm{i}}}\right)\sqrt{\frac{2H}{g}}$. The minimum value of the function $\Delta x_{\mathrm{total}}$ occurs for $\frac{d(\Delta x_{\mathrm{total}})}{d(\Delta x_{\mathrm{i}})} = 0$ or

$1 - \frac{\hbar}{2m}\sqrt{\frac{2H}{g}}(\Delta x_{\mathrm{i}})^{-2} = 0$.



We find $\Delta x_{\mathrm{i}} = \left(\frac{\hbar}{2m}\right)^{1/2}\left(\frac{2H}{g}\right)^{1/4}$.

(b) For $H = 2.0\ \mathrm{m}$, $m = 0.50\ \mathrm{g}$, $\Delta x_{\mathrm{total}} = 5.2 \times 10^{-16}\ \mathrm{m}$.

5-27 For a single slit with width a, minima are given by $\sin\theta = \frac{n\lambda}{a}$ where $n = 1, 2, 3, \ldots$ and $\sin\theta \approx \tan\theta = \frac{x}{L}$, $\frac{x_1}{L} = \frac{\lambda}{a}$ and $\frac{x_2}{L} = \frac{2\lambda}{a} \Rightarrow \frac{x_2 - x_1}{L} = \frac{\lambda}{a}$ or

$$\lambda = \frac{a\Delta x}{L} = \frac{5\ \text{Å} \times 2.1\ \text{cm}}{20\ \text{cm}} = 0.525\ \text{Å}$$

$$E = \frac{p^2}{2m} = \frac{h^2}{2m\lambda^2} = \frac{(hc)^2}{2mc^2\lambda^2} = \frac{\left(1.24 \times 10^4\ \text{eV}\cdot\text{Å}\right)^2}{2\left(5.11 \times 10^5\ \text{eV}\right)\left(0.525\ \text{Å}\right)^2} = 546\ \text{eV}$$

5-29

With one slit open, $P_1 = |\Psi_1|^2$ or $P_2 = |\Psi_2|^2$.

Ψ_1 is the wavefunction of the electron after it passes through slit 1, and sim. for Ψ_2.

With both slits open, $P = |\Psi_1 + \Psi_2|^2$ → The wavefn's just add (that's superposition!).

At a max, these guys reinforce: $P_{max} = (|\Psi_1| + |\Psi_2|)^2$

At a min, they counteract: $P_{min} = (|\Psi_1| - |\Psi_2|)^2$

We know that $\frac{P_1}{P_2} = 25 = \frac{|\Psi_1|^2}{|\Psi_2|^2} \Rightarrow \frac{|\Psi_1|}{|\Psi_2|} = 5 \Rightarrow |\Psi_1| = 5|\Psi_2|$

$$\text{So } \frac{P_{max}}{P_{min}} = \frac{|\Psi_1|^2 + 2|\Psi_1||\Psi_2| + |\Psi_2|^2}{|\Psi_1|^2 - 2|\Psi_1||\Psi_2| + |\Psi_2|^2}$$

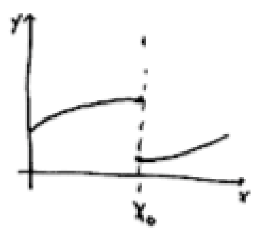
$$= \frac{25+10+1}{25-10+1} = \boxed{\frac{36}{16}}$$

#1.) Of the functions graphed in Figure P5.1, which are candidates for the Schrödinger wavefunction of an actual physical system? For those that are not, state why they fail to qualify.

There's only a few absolute requirements for a wavefunction:

1.) Ψ represents a probability for locating a particle at a point, and should therefore be <u>single-valued</u> for any given x. It doesn't make any sense for there to be different values of the probability located at the same point.

2.) This implies that Ψ should also be <u>continuous</u>; no big ugly jumps. Something discontinuous, like

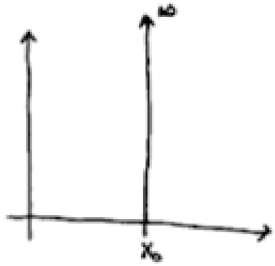


Is double valued at x_0, if we examine

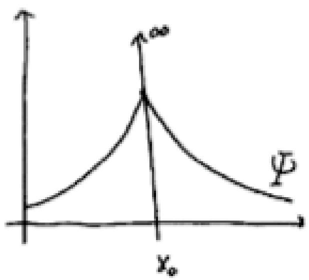
$$\lim_{x \to x_0^+} \psi \neq \lim_{x \to x_0^-} \psi$$

3.) Most of the time, we will also require Ψ to be <u>smooth</u>, that is, require $\frac{d\Psi}{dx}$ to be continuous.

This can be violated, though, for example near infinite potentials.
For example, something called a "delta function potential"
looks like a single infinite spike at some point x_0



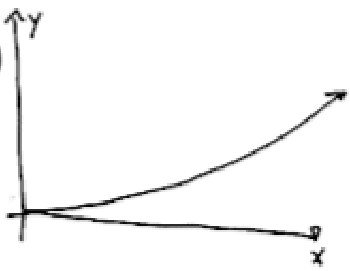
At x_0, Ψ won't be smooth, but it will be continuous and single valued:



4.) Last, we require Ψ to be finite for <u>all</u> values of x (even $x \to \pm\infty$). No infinite probabilities!

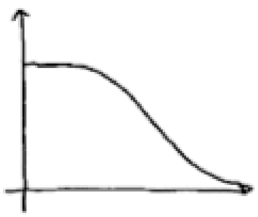
Down to business

a.)



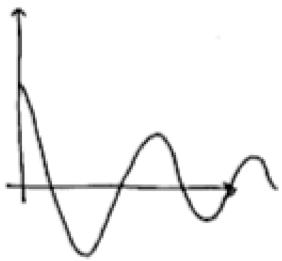
No good! As $x \to \infty$ $\Psi \to \infty$, so it violates Kuhlman golden rule #4.

b.)



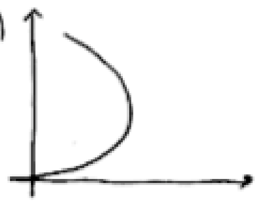
Nice, smooth, finite, and continuous.
Looks good!

c.)



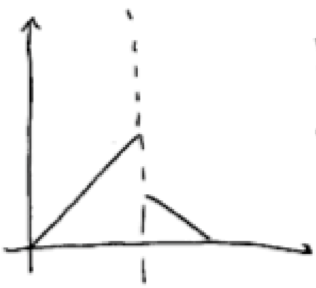
Even looks wavy! What more could you ask for?

d.)



Stinky! Double valued for some x.
No go.

e.)



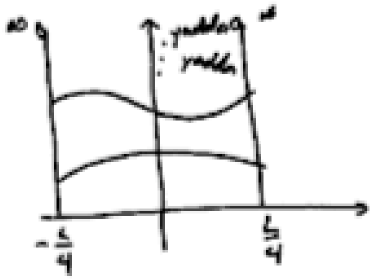
Even worse! At least d.) was continuous
Letter e.), you suck.

A particle is described by the wavefunction

$$\psi(x) = \begin{cases} A\cos\left(\frac{2\pi x}{L}\right) & -\frac{L}{4} \le x \le \frac{L}{4} \\ 0 & \text{otherwise} \end{cases}$$

a.) Determine the normalization constant A.

First of all, this is our old friend particle in a box, where the box is length $\frac{L}{2}$



So we could just cheat and say "We found that in section 5.4. Let's use that and let $L \to \frac{L}{2}$!"

This way is for wussies! Let's tough it out so we can look ourselves in the mirror.

$|\psi(x)|^2$ is the probability amplitude, and to find the probability to find a particle in the region $a \le x \le b$,

$$P = \int_a^b |\psi(x)|^2\,dx$$

Now, our particle is inside this box, so we know

$$P = \int_{-\frac{L}{4}}^{\frac{L}{4}} |\psi(x)|^2\,dx$$

It <u>HAS</u> to be in the box somewhere! So the probability to locate it in there is 100%

$$P = \int_{-\frac{L}{4}}^{\frac{L}{4}} |\psi(x)|^2 \, dx = 1$$

$$= \int_{-\frac{L}{4}}^{\frac{L}{4}} A^2 \cos^2\left(\frac{2\pi x}{L}\right) dx$$

$$= A^2 \int_{-\frac{L}{4}}^{\frac{L}{4}} \cos^2\left(\frac{2\pi x}{L}\right) dx$$

Let $u = \frac{2\pi x}{L}$ when $x = -\frac{L}{4}$, $u = -\frac{\pi}{2}$

$du = \frac{2\pi}{L} dx$ $x = \frac{L}{4}$, $u = \frac{\pi}{2}$

$\Rightarrow dx = \frac{L}{2\pi} du$

$$= \frac{A^2 L}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 u \, du$$

Remember how to do this one?

$$= \frac{A^2 L}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}(1 + \cos 2u) \, du$$

$$= \frac{A^2 L}{2\pi} \left[\frac{u}{2} + \frac{\sin 2u}{4} \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{A^2 L}{4}$$

And so we know

$$\frac{A^2 L}{4} = 1$$

$$\therefore \boxed{A = \sqrt{\frac{4}{L}}}$$

b) What is the probability that the particle will be found between $x=0$ and $x=\frac{L}{8}$ if a measurement of its position is made?

Just do

$$P = \int_0^{\frac{L}{8}} |\psi(x)|^2 dx$$

$$= \frac{4}{L}\int_0^{\frac{L}{8}} \cos^2\left(\frac{2\pi x}{L}\right) dx$$

Do this the same way as part a.)

$$= \frac{2}{\pi}\int_0^{\frac{\pi}{4}} \cos^2 u \, du$$

$$= \frac{2}{\pi}\left[\frac{u}{2} + \frac{\sin 2u}{4}\right]\Bigg|_0^{\frac{\pi}{4}}$$

$$\boxed{P = \frac{2}{\pi}\left(\frac{\pi}{8} + \frac{1}{4}\right)}$$

6-3

A free electron has a wavefunction

$$\psi(x) = A\sin(5\times10^{10}\, x)$$

where x is measured in meters (That means 5×10^{10} has units m^{-1}!) Find

a.) the electron's de Broglie wavelength

For a free particle, the wavefunction <u>always</u> has the form

$$\psi(x) = A\sin(kx + \phi)$$

So we know that in our case, $\phi = 0$ and

$$k = 5\times10^{10} = \frac{2\pi}{\lambda}$$

$$\therefore \boxed{\lambda = \frac{2\pi}{5\times10^{10}} = 0.126 \text{ nm}}$$

b.) the electron's momentum

Well, you should remember

$$\lambda = \frac{h}{p} \Rightarrow \boxed{p = \frac{h}{\lambda} = 5.26\times10^{-24}\ \frac{kg\,m}{s} = 32.8\ \mu\frac{eV}{c}}$$

That's a little sissy momentum. Using $p = mv$ gives you $v = 0.01\,c$, so we're nonrelativistic

c.) the electron's energy in electron volts

$$\boxed{K = \frac{p^2}{2m} = 15.1\times10^{-18}\text{ J} = 94\text{ eV}}$$

6-4 The time development of Ψ is given by Equation 6.8 or

$$\Psi(x,\ t) = \int a(k) e^{i\{kx-\omega(k)t\}} dk = \left(\frac{C\alpha}{\sqrt{\pi}}\right)\int_{-\infty}^{\infty} e^{\{ikx - i\omega(k)t - \alpha^2 k^2\}} dk,$$

with $\omega(k) = \frac{\hbar k^2}{2m}$ for a free particle of mass m. As in Example 6.3, the integral may be reduced to a recognizable form by completing the square in the exponent. Since $\omega(k)t = \left(\frac{\hbar t}{2m}\right)k^2$, we group this term together with $\alpha^2 k^2$ by introducing

$$\beta^2 = \alpha^2 + \frac{i\hbar t}{2m} \text{ to get}$$

$$ikx - \omega(k)t - \alpha^2 k^2 = -\left(\beta k - \frac{ix}{2\beta}\right)^2 - \frac{x^2}{4\beta^2}.$$

Then, changing variables to $z = \beta k - \frac{ix}{2\beta}$ gives

$$\Psi(x,\ t) = \left(\frac{C\alpha}{\beta\sqrt{\pi}}\right) e^{-x^2/4\beta^2} \int_{-\infty}^{\infty} e^{-z^2} = \left(\frac{C\alpha}{\beta}\right) e^{-x^2/4\beta^2}.$$

To interpret this result, we must recognize that β is complex and separate real and imaginary parts. Thus, $\left|\beta^2\right|^2 = \left|\alpha^2 + \frac{i\hbar t}{2m}\right|^2 = \alpha^4 + \left(\frac{\hbar t}{2m}\right)^2$ and the exponent for Ψ is

$$\frac{x^2}{4\beta^2} = \frac{x^2\left(\alpha^2 - \frac{i\hbar t}{2m}\right)}{4\left|\beta^2\right|^2} = \frac{x^2}{4\left[\alpha^2 + \left(\frac{\hbar t}{2m\alpha}\right)^2\right]} + (\text{imaginary terms})$$

then

$$\left|\Psi(x,\ t)\right| = \frac{C\alpha}{\left(\alpha^4 + \left(\frac{\hbar t}{2m}\right)^2\right)^{1/4}} e^{\left\{-x^2/\left[4\left\{\alpha^2 + (\hbar t/2m\alpha)^2\right\}\right]\right\}}.$$

We see that apart from a phase factor, $\Psi(x,\ t)$ is still a gaussian but with amplitude diminished by $\frac{\alpha}{\left(\alpha^4 + \left(\frac{\hbar t}{2m}\right)^2\right)^{1/4}}$ and a width $\Delta x(t) = \left(\alpha^2 + \left(\frac{\hbar t}{2m\alpha}\right)^2\right)^{1/2}$ where $\alpha = \Delta x(0)$ is the initial width.

6-5

In a region of space, a particle with zero energy has a wave function

$$\psi(x) = A x e^{-x^2/L^2}$$

a.) Find the potential energy U as a function of x.

The Schrödinger equation has the form

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + U(x)\psi(x) = E\psi(x)$$

For our particle, E=0, so

$$\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} = U(x)\psi(x)$$

$$\Rightarrow U(x) = \frac{1}{\psi(x)}\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2}$$

Right? Plug in our wavefunction.

$$U(x) = \frac{1}{Axe^{-x^2/L^2}} \frac{\hbar^2}{2m} \frac{d^2}{dx^2}\left(Axe^{-x^2/L^2}\right)$$

Take Those derivatives! (Don't forget chain + product rule!)

$$\frac{d^2}{dx^2}\left(Axe^{-x^2/L^2}\right) = \frac{d}{dx}\left(Ae^{-x^2/L^2} - \frac{2A}{L^2}x^2e^{-x^2/L^2}\right)$$

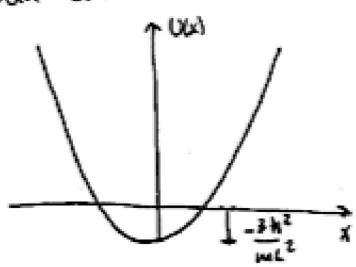
$$= -\frac{2Ax}{L^2}e^{-x^2/L^2} - \frac{4Ax}{L^2}e^{-x^2/L^2} + \frac{4Ax^3}{L^4}e^{-x^2/L^2}$$

$$= \frac{2Ax(2x^2-3L^2)e^{-x^2/L^2}}{L^4}$$

So $\boxed{U(x) = \frac{\hbar^2}{2mL^2}\left(\frac{4}{L^2}x^2 - 6\right)}$

b.) Make a sketch of U(x) versus x

Parabola centered at $x=0$ with $U(0) = -\frac{3\hbar^2}{mL^2}$



6-6 $\psi(x) = A\cos kx + B\sin kx$

$$\frac{\partial\psi}{\partial x} = -kA\sin kx + kB\cos kx$$

$$\frac{\partial^2\psi}{\partial x^2} = -k^2A\cos kx - k^2B\sin kx$$

$$\left(\frac{-2m}{\hbar^2}\right)(E-U)\psi = \left(\frac{-2mE}{\hbar^2}\right)(A\cos kx + B\sin kx)$$

The Schrödinger equation is satisfied if $\frac{\partial^2\psi}{\partial x^2} = \left(\frac{-2m}{\hbar^2}\right)(E-U)\psi$ or

$$-k^2(A\cos kx + B\sin kx) = \left(\frac{-2mE}{\hbar^2}\right)(A\cos kx + B\sin kx).$$

Therefore $E = \frac{\hbar^2 k^2}{2m}$.

6-7

Show that allowing the state $n=0$ for a particle in a one dimensional box violates the uncertainty principle, $\Delta x \cdot \Delta p \geq \frac{\hbar}{2}$

The particle is somewhere inside the box, right?

So $\Delta x = L$ (L = box length)

For a particle in a box,

$$E_n = \frac{n^2 \pi^2 \hbar^2}{2mL^2}$$ (Eq. 5.17)

for $n=0$

$E_n = 0$

Inside the box, all energy is kinetic

$$E = K = \frac{\langle p^2 \rangle}{2m} = 0$$

$$\therefore \langle p^2 \rangle = 0$$

Now,

$$\psi_{(x)} = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$$

So that

$$\langle p \rangle = \int_0^L \psi^*_{(x)} \, \hat{p} \, \psi_{(x)} \, dx$$

$$= \frac{2}{L} \int_0^L \sin\left(\frac{n\pi x}{L}\right) \left(-i\hbar \frac{d}{dx}\right) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$= -i\hbar \left(\frac{2n\pi}{L^2}\right) \int_0^L \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx$$

$$= 0$$

So that

$$\Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} \qquad \text{Eq. 5.34}$$

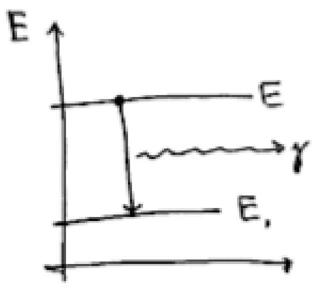
$$= 0$$

$$\therefore \boxed{\Delta x \, \Delta p = 0}$$

6-9

This is easier than it sounds. We are describing the situation as a particle in a box so we know the energy states are

$$E_n = \frac{n^2\pi^2\hbar^2}{2m_pL^2}$$



So $E_2 = \frac{4\pi^2\hbar^2}{2m_pL^2}$

$$E_1 = \frac{\pi^2\hbar^2}{2m_pL^2}$$

The emitted photon has the difference in energy between these two states

$$\boxed{\Delta E = \frac{3\pi^2\hbar^2}{2m_pL^2} = 6.14\text{ MeV}}$$

$$\Delta E = \frac{hc}{\lambda} \Rightarrow \boxed{\lambda = \frac{hc}{\Delta E} = 2.02\times10^{-4}\text{ nm}}$$

That's in the gamma ray region

6-10 $E_n = \frac{n^2h^2}{8mL^2}$

$$\frac{h^2}{8mL^2} = \frac{\left(6.63\times10^{-34}\text{ Js}\right)^2}{8\left(9.11\times10^{-31}\text{ kg}\right)\left(10^{-10}\text{ m}\right)^2} = 6.03\times10^{-18}\text{ J} = 37.7\text{ eV}$$

(a) $E_1 = 37.7\text{ eV}$

$E_2 = 37.7\times2^2 = 151\text{ eV}$

$E_3 = 37.7\times3^2 = 339\text{ eV}$

$E_4 = 37.7\times4^2 = 603\text{ eV}$

(b) $hf = \frac{hc}{\lambda} = E_{n_i} - E_{n_f}$

$$\lambda = \frac{hc}{E_{n_i} - E_{n_f}} = \frac{1\,240 \text{ eV} \cdot \text{nm}}{E_{n_i} - E_{n_f}}$$

For $n_i = 4$, $n_f = 1$, $E_{n_i} - E_{n_f} = 603 \text{ eV} - 37.7 \text{ eV} = 565 \text{ eV}$, $\lambda = 2.19 \text{ nm}$

$n_i = 4$, $n_f = 2$, $\lambda = 2.75 \text{ nm}$

$n_i = 4$, $n_f = 3$, $\lambda = 4.70 \text{ nm}$

$n_i = 3$, $n_f = 1$, $\lambda = 4.12 \text{ nm}$

$n_i = 3$, $n_f = 2$, $\lambda = 6.59 \text{ nm}$

$n_i = 2$, $n_f = 1$, $\lambda = 10.9 \text{ nm}$