Physics 2D, Winter 2005
Week 2 Exercise Solutions: Relativity

See last week's solutions for problems 1-14 through 1-18.

1-20 $$u = \frac{v + u'}{1 + vu'/c^2} = \frac{0.90c + 0.70c}{1 + (0.90c)(0.70c)/c^2} = 0.98c$$

1-22 (a) The speed as observed in the laboratory is found by using Equation 1.30:
$$u_x = \frac{u'_x + v}{1 + u'_x v/c^2}.$$
But $u'_X = \dfrac{c}{n}$ (speed measured by an observer moving with the fluid), therefore
$$u_X = \frac{(c/n) + v}{1 + v/(nc)} = \frac{c}{n}\,\frac{1 + nv/c}{1 + v/(nc)}.$$

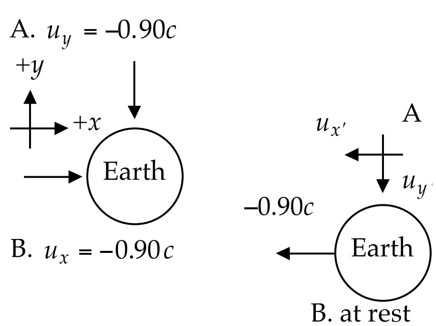
(b) $\dfrac{v}{c} << 1$. Use the binomial expansion,
$$u'_X \cong \frac{c}{n}\left[1 + n\left(\frac{v}{c}\right)\right]\left[1 - \frac{v}{nc}\right] \cong \frac{c}{n}\left[1 + \frac{nv}{c} - \frac{v^2}{c^2}\right] \cong \frac{c}{n} + v - \frac{v}{n^2}.$$

1-23 (a) In S', $x'_1 = \gamma(x_1 - vt_1) = 0$, $y'_1 = y_1 = 0$, $z'_1 = z_1 = 0$, and $t'_1 = \gamma\left[t_1 - \left(\dfrac{v}{c^2}\right)x_1\right] = 0$, with $\gamma = \left[1 - \dfrac{v^2}{c^2}\right]^{-1/2}$ and so $\gamma = \left[1 - (0.70)^2\right]^{-1/2} = 1.40$. In system S', $x'_2 = \gamma(x_2 - vt_2) = 140\text{ m}$, $y'_2 = z'_2 = 0$, and
$$t'_2 = \gamma\left[t_2 - \left(\frac{v}{c^2}\right)x_2\right] = \frac{(1.4)(-0.70)(100\text{ m})}{3.00 \times 10^8\text{ m/s}} = -0.33\ \mu\text{s}.$$

(b) $\Delta x' = x'_2 - x'_1 = 140\text{ m}$

(c) Events are not simultaneous in S', event 2 occurs $0.33\ \mu\text{s}$ earlier than event 1.

1-24



Use the velocity of the Earth as seen by the pilot of Spaceship B, for v, and use the velocity of Spaceship A as seen from Earth, for $\boldsymbol{u}$.

$$u'_x = \frac{u_x - v}{1 - u_x v / c^2} = \frac{0 - 0.90c}{1 - (0)(0.90c)/c^2} = -0.90c$$

$$u'_y = \frac{u_y}{\gamma\left(1 - u_x v / c^2\right)} = \frac{0 - 0.90c}{\left[1 - 0.81\right]^{-1/2}} \cong -0.392c$$

The speed of A as measured by B is

$$u_{AB} = \left[\left(u'_x\right)^2 + \left(u'_y\right)^2\right]^{1/2} = \left[(-0.90c)^2 + (-0.392c)^2\right]^{1/2} = 0.982c\,.$$

Classically, $u_{AB} = 1.3c$. How absurd! Remember that we're only looking for the speed, so don't get hung up on the signs of $\boldsymbol{u}'$.

1-33 (a) First, note that when the text states "we see … explode", it means, "light from … exploding reaches us." We in the spaceship do not calculate the explosions to be simultaneous. We measure the distance we have traveled from the Sun as

$$L = L_p\sqrt{1 - \left(\frac{v}{c}\right)^2} = (6.00\text{ ly})\sqrt{1 - (0.800)^2} = 3.60\text{ ly}\,.$$

We see the Sun flying away from us at $0.800c$ while the light from the Sun approaches at $1.00c$. Thus, the gap between the Sun and its blast wave has opened at $1.80c$, and the time we calculate to have elapsed since the Sun exploded is $\frac{3.60\text{ ly}}{1.80c} = 2.00\text{ yr}$. We see Tau Ceti as moving toward us at $0.800c$, while its light approaches at $1.00c$, only $0.200c$ faster. We measure the gap between that star and its blast wave as 3.60 ly and growing at $0.200c$. We calculate that it must have been opening for $\frac{3.60\text{ ly}}{0.200c} = 18.0\text{ yr}$ and conclude that $\boxed{\text{Tau Ceti exploded 16.0 years before the Sun}}$.

(b) Consider a hermit who lives on an asteroid halfway between the Sun and Tau Ceti, stationary with respect to both. Just as our spaceship is passing him, he also sees the blast waves from both explosions. Judging both stars to be stationary, this observer concludes that $\boxed{\text{the two stars blew up simultaneously}}$.

1-39 (a) For the satellite $\sum F = ma$: $\frac{GM_E m}{r^2} = \frac{mv^2}{r} = \frac{m}{r}\left(\frac{2\pi r}{T}\right)^2$

$$GM_E T^2 = 4\pi^2 r^3$$

$$r = \left(\frac{6.67\times10^{-11}\text{ N}\cdot\text{m}^2\left(5.98\times10^{24}\text{ kg}\right)\left(43\,080\text{ s}\right)^2}{\text{kg}^2\,4\pi^2}\right)^{1/3} = 2.66\times10^{7}\text{ m}$$

(b) $$v = \frac{2\pi r}{T} = \frac{2\pi\left(2.66\times10^{7}\text{ m}\right)}{43\,080\text{ s}} = 3.87\times10^{3}\text{ m/s}$$

(c) The small fractional decrease in frequency received is equal in magnitude to the fractional increase in period of the moving oscillator due to time dilation:

$$\text{fractional change in } f = -(\gamma - 1) = -\left[\frac{1}{\sqrt{1-\left(3.87\times 10^{3}/3\times 10^{8}\right)^{2}}} - 1\right]$$

$$= 1 - \left(1 - \frac{1}{2}\left[-\left(\frac{3.87\times 10^{3}}{3\times 10^{8}}\right)^{2}\right]\right) = -8.34\times 10^{-11}$$

(d) The orbit altitude is large compared to the radius of the Earth, so we must use $U_g = -\frac{GM_E m}{r}$.

$$\Delta U_g = -\frac{6.67\times 10^{-11}\ \mathrm{N\cdot m^2}\left(5.98\times 10^{24}\ \mathrm{kg}\right)m}{\mathrm{kg^2}\ 2.66\times 10^{7}\ \mathrm{m}} + \frac{6.67\times 10^{-11}\ \mathrm{N\cdot m^2}\left(5.98\times 10^{24}\ \mathrm{kg}\right)m}{\mathrm{kg^2}\ 6.37\times 10^{6}\ \mathrm{m}}$$

$$= 4.76\times 10^{7}\ \mathrm{J/kg}\ m$$

$$\frac{\Delta f}{f} = \frac{\Delta U_g}{mc^2} = \frac{4.76\times 10^{7}\ \mathrm{m^2/s^2}}{\left(3\times 10^{8}\ \mathrm{m/s}\right)^2} = +5.29\times 10^{-10}$$

(e) $-8.34\times 10^{-11} + 5.29\times 10^{-10} = +4.46\times 10^{-10}$

2-4 (a) As **F** is parallel to **v**, scalar equations are used. Relativistic momentum is given by $p = \gamma mv = \frac{mv}{\left[1-(v/c)^2\right]^{1/2}}$, and relativistic force is given by

$$F = \frac{dp}{dt} = \frac{d}{dt}\left\{\frac{mv}{\left[1-(v/c)^2\right]^{1/2}}\right\}$$

$$F = \frac{dp}{dt} = \frac{m}{\left[1-\left(v^2/c^2\right)\right]^{3/2}}\left(\frac{dv}{dt}\right)$$

Hence, $qE = m\left(1-\frac{v^2}{c^2}\right)^{-3/2}\left(\frac{dv}{dt}\right)$, or $a = \frac{dv}{dt} = \left(\frac{qE}{m}\right)\left(1-\frac{v^2}{c^2}\right)^{3/2}$. Here v is a function of t and q, E, M, and c are parameters.

(b) As expected; as $v \to c$, $a \to 0$ because in general no speed can exceed c, the speed of light.

(c) Separating variables, $\frac{dv}{\left(1-v^2/c^2\right)^{3/2}} = \left(\frac{qE}{m}\right)dt$, or $\int_0^v \frac{dv}{\left(1-v^2/c^2\right)^{3/2}} = \int_0^t \frac{qE}{m}\,dt$ (note that the integral on the left is the opposite of the differentiation in part (a)),

$$\left.\frac{v}{\left(1-v^2/c^2\right)^{1/2}}\right|_0^v = \frac{qEt}{m}$$

$$\frac{v^2}{\left(1-v^2/c^2\right)^{1/2}} = \left(\frac{qEt}{m}\right)^2 = \frac{qEt}{m}$$

$$v^2 = \left(\frac{qEt}{m}\right)^2 - \left(\frac{v^2}{c^2}\right)\left(\frac{qEt}{m}\right)^2$$

$$v^2\left[1+\left(\frac{qEt}{m}\right)^2\right] = \left(\frac{qEt}{m}\right)^2$$

$$v^2 = \frac{\left(qEt/mc\right)^2}{1+\left(qEt/mc\right)^2}$$

$$v^2 = \frac{\left(qEct\right)^2}{\left(mc\right)^2+\left(qEt\right)^2}$$

Note that the limiting behavior of v as $t \to 0$ and $t \to \infty$ is reasonable. As

$$v = \frac{dx}{dt} = \frac{qEct}{\left[\left(mc\right)^2+\left(qEt\right)^2\right]^{1/2}},$$

$$x = qEc\left[\left(mc\right)^2+\left(qEt\right)^2\right]^{1/2}\left.\left[\frac{1}{\left(qE\right)^2}\right]\right|_0^t = \frac{c}{qE}\left\{\left[\left(mc\right)^2+\left(qEt\right)^2\right]^{1/2} - mc\right\}.$$

As $t \to 0$, $x \to 0$, and $t \to \infty$, $x \to ct$; reasonable results.

2-12 (a) When $K_e = K_p$, $m_e c^2\left(\gamma_e - 1\right) = m_p c^2\left(\gamma_p - 1\right)$. In this case $m_e c^2 = 0.511\,0\text{ MeV}$ and $m_p c^2 = 938\text{ MeV}$, $\gamma_e = \left[1-\left(0.75\right)^2\right]^{1/2} = 1.511\,9$. Substituting these values into the first equation, we find $\gamma_p = 1 + \frac{m_e c^2\left(\gamma - 1\right)}{m_p c^2} = 1 + \frac{\left(0.511\,0\right)\left(1.511\,9 - 1\right)}{939} = 1.000\,279$. But $\gamma_p = \frac{1}{\left[1-\left(u_p/c\right)^2\right]^{1/2}}$; therefore $u_p = c\left(1-\gamma_p^{-2}\right)^{1/2} = 0.023\,6c$.

(b) When $p_e = p_p$, $\gamma_p m_p u_p = \gamma_e m_e u_e$

$$\frac{u_p}{\sqrt{1-(u_p/c)^2}} = \gamma_e \frac{m_e}{m_p} u_e$$

$$u_p{}^2 = \left(\gamma_e \frac{m_e}{m_p} u_e\right)^2\left(1 - \frac{u_p{}^2}{c^2}\right)$$

$$u_p = \frac{\gamma_e \frac{m_e}{m_p} u_e}{\sqrt{1+\left(\gamma_e \frac{m_e}{m_p} \frac{u_e}{c^2}\right)^2}} = \frac{(1.5119)\left(\frac{0.511}{939}\right)(0.75\,c)}{\sqrt{1+\left[(1.5119)\left(\frac{0.511}{939}\right)(0.75)\right]^2}} \approx 6.7\times10^{-4}\,c$$

Notice that using $\gamma_p = 1$ in these calculations has a very small effect and makes the equations much easier to solve! Try it and see for yourself.

2-14 (a) $E = mc^2$

$$m = \frac{E}{c^2} = \frac{4 \times 10^{26}\ \text{J}}{\left(3.0 \times 10^8\ \text{m/s}\right)^2} = 4.4 \times 10^9\ \text{kg}$$

(b) $$t = \frac{\left(2.0 \times 10^{30}\right)\ \text{kg}}{4.4 \times 10^9\ \text{kg/s}} = 4.5 \times 10^{20}\ \text{s} = 1.4 \times 10^{13}\ \text{years}$$

2-16 (a) In S the speed of the particle is u so $p = \dfrac{mu}{(1 - u^2/c^2)^{1/2}}$, $E = \dfrac{mc^2}{\left(1 - u^2/c^2\right)^{1/2}}$, and

$\left(E^2 - p^2c^2\right) = m^2c^4$.

In S', $u' = \dfrac{u - v}{1 - uv/c^2}$,

$\gamma' = 1 \Big/ \sqrt{1 - [(u - v)/(1 - uv/c^2)]^2/c^2}$

$p' = m(u - v)/(1 - uv/c^2)\gamma'$

$E' = mc^2\gamma'$

(b) Using these expressions for E' and p', one obtains $\left(E'^2 - p'^2c^2\right) = m^2c^4$, just as for the unprimed frame.