Physics 2D, Winter 2005
Week 1 Exercise Solutions: Relativity

1-2 **IN THE REST FRAME:**

In an elastic collision, energy and momentum are conserved.

$$p_i = m_1 v_{1i} + m_2 v_{2i} = p_f = m_1 v_{1f} + m_2 v_{2f}$$

or $m_1(v_{1i} - v_{1f}) = -m_2(v_{2i} - v_{2f})$. The energy equation is

$$E_i = E_f = \frac{1}{2}m_1 v_{1i}^2 + \frac{1}{2}m_2 v_{2i}^2 = \frac{1}{2}m_1 v_{1f}^2 + \frac{1}{2}m_2 v_{2f}^2$$

or $m_1(v_{1i} - v_{1f})(v_{1i} + v_{1f}) = -m_2(v_{2i} - v_{2f})(v_{2i} + v_{2f})$. Substituting the momentum equation into the energy equation yields a very simple and general result (true even for three-dimensional collision if the velocities are replaced as vectors) $(v_{1i} + v_{1f}) = (v_{2i} + v_{2f})$ or as Newton put it originally, the final relative velocity is opposite to the initial relative velocity:

$$(v_{1i} - v_{2i}) = -(v_{1f} - v_{2f})$$

for elastic collision (and a fraction of the initial for general collisions). Now, putting in the numerical values, the momentum equation, and this relative velocity equation gives: $m_1 v_{1f} + m_2 v_{2f} = 0.3 v_{1f} + 0.2 v_{2f} = 0.9 \text{ kg}\cdot\text{m/s}$, and $(5 - (-3)) = -(v_{1f} - v_{2f})$. Solving the two equations, two unknowns, we find $v_{1f} = -1.4 \text{ m/s}$ and $v_{2f} = +6.6 \text{ m/s}$.

IN THE MOVING FRAME:

The Galilean velocity transformations hold.

$$v'_{1i} - v' = 20 \text{ m/s} - 10 \text{ m/s} = 10 \text{ m/s}$$
$$v'_{2i} = v_{2i} - v' = 0 \text{ m/s} - 10 \text{ m/s} = -10 \text{ m/s}$$
$$p'_i = m_1 v'_{1i} + m_2 v'_{2i} = (2\,000 \text{ kg})(10 \text{ m/s}) - (1\,500 \text{ kg})(10 \text{ m/s}) = 5 \times 10^3 \text{ kg}\cdot\text{m/s}$$
$$p'_f = (2\,000 \text{ kg} + 1\,500 \text{ kg})v'_f = (3\,500 \text{ kg})(v_f - 10 \text{ m/s}), \text{ and because } v_f = 11.4 \text{ m/s},$$
$$p'_f = 5 \times 10^3 \text{ kg}\cdot\text{m/s}$$

1-3 **IN THE REST FRAME:**

In an elastic collision energy and momentum are conserved.

$$p_i = m_1 v_{1i} + m_2 v_{2i} = (0.3 \text{ kg})(5 \text{ m/s}) + (0.2 \text{ kg})(-3 \text{ m/s}) = 0.9 \text{ kg}\cdot\text{m/s}$$
$$p_f = m_1 v_{1f} + m_2 v_{2f}$$

This equation has two unknowns, therefore, apply the conservation of kinetic energy $E_i = E_f = \frac{1}{2}m_1 v_{1i}^2 + \frac{1}{2}m_2 v_{2i}^2 = \frac{1}{2}m_1 v_{1f}^2 + \frac{1}{2}m_2 v_{2f}^2$ and conservation of momentum one finds that $v_{1f} = -1.31 \text{ m/s}$ and $v_{2f} = 6.47 \text{ m/s}$ or $v_{1f} = -1.56 \text{ m/s}$ and $v_{2f} = 6.38 \text{ m/s}$. The difference in values is due to the rounding off errors in the numerical calculations of the mathematical quantities. If these two values are averaged the values are $v_{1f} = -1.4 \text{ m/s}$ and $v_{2f} = 6.6 \text{ m/s}$, $p_f = 0.9 \text{ kg}\cdot\text{m/s}$. Thus, $p_i = p_f$.

IN THE MOVING FRAME:
Make use of the Galilean velocity transformation equations. $p_i' = m_1 v_{1i}' + m_2 v_{2i}'$; where $v_{1i}' = v_{1i} - v' = 5\ \text{m/s} - (-2\ \text{m/s}) = 7\ \text{m/s}$. Similarly, $v_{2i}' = -1\ \text{m/s}$ and $p_i' = 1.9\ \text{kg}\cdot\text{m/s}$. To find p_f' use $v_{1f}' = v_{1i} - v'$ and $v_{2f}' = v_{2i} - v'$ because the prime system is now moving to the left. Using these results give $p_f' = 1.9\ \text{kg}\cdot\text{m/s}$.

1-4 (a) In all cases one wants the speed of the plane relative to the ground. For the upwind and downwind legs, where v' in the figure is given by $(c^2 - v^2)^{1/2}$

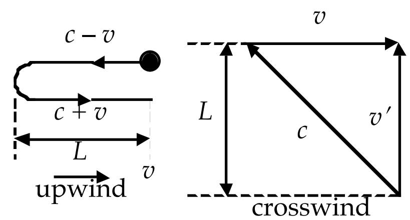
$$t_{u+d} = \frac{L}{c-v} + \frac{L}{c+v} = \frac{2L}{c}\left(\frac{1}{1 - v^2/c^2}\right).$$

For the crosswind case, the plane's speed along L is $v' = (c^2 - v^2)^{1/2}$

$$t_c = \frac{2L}{\sqrt{c^2 - v^2}} = \frac{2L}{c}\frac{1}{\sqrt{1 - (v/c)^2}}$$

$$t_{u+d} = \frac{2(100\ \text{mi})}{500\ \text{mi/h}}\left(\frac{1}{1 - (100)^2/(500)^2}\right) = 0.416\,7\ \text{h}$$

$$t_c = \frac{2(100\ \text{mi})}{500\ \text{mi/h}}\left(\frac{1}{\sqrt{0.96}}\right) = 0.408\,2\ \text{h}$$



(b) $\Delta t = t_{u+d} - t_c = 0.008\,5\ \text{h} \cong 0.009\ \text{h}$ or $0.510\ \text{min} \cong 0.5\ \text{min}$

1-5 This is a case of dilation. $T = \gamma T'$ in this problem with the proper time $T' = T_0$

$$T = \left[1 - \left(\frac{v}{c}\right)^2\right]^{-1/2} T_0 \Rightarrow \frac{v}{c} = \left[1 - \left(\frac{T_0}{T}\right)^2\right]^{1/2};$$

in this case $T = 2T_0$, $v = \left\{1 - \left[\frac{L_0/2}{L_0}\right]^2\right\}^{1/2} = \left[1 - \left(\frac{1}{4}\right)\right]^{1/2}$ therefore $v = 0.866c$.

1-6 This is a case of length contraction. $L = \frac{L'}{\gamma}$ in this problem the proper length $L' = L_0$,

$L = \left[1 - \frac{v^2}{c^2}\right]^{-1/2} L_0 \Rightarrow v = c\left[1 - \left(\frac{L}{L_0}\right)^2\right]^{1/2}$; in this case $L = \frac{L_0}{2}$, $v = \left\{1 - \left[\frac{L_0/2}{L_0}\right]^2\right\}^{1/2} = \left[1 - \left(\frac{1}{4}\right)\right]^{1/2}$ therefore $v = 0.866c$.

1-7 The problem is solved by using time dilation. This is also a case of $v \ll c$ so the binomial expansion is used $\Delta t = \gamma\, \Delta t' \cong \left[1 + \frac{v^2}{2c^2}\right]\Delta t'$, $\Delta t - \Delta t' = \frac{v^2 \Delta t'}{2c^2}$; $v = \left[\frac{2c^2(\Delta t - \Delta t')}{\Delta t'}\right]^{1/2}$;

$\Delta t = (24\ \text{h/day})(3\,600\ \text{s/h}) = 86\,400\ \text{s}$; $\Delta t' = \Delta t - 1 = 86\,399\ \text{s}$;

$$v = \left[\frac{2\left(86\,400\ \text{s} - 86\,399\ \text{s}\right)}{86\,399\ \text{s}}\right]^{1/2} = 0.004\,8c = 1.44\times 10^{6}\ \text{m/s}.$$

1-8 $L = \dfrac{L'}{\gamma}$

$$\frac{1}{\gamma} = \frac{L}{L'} = \left[1 - \frac{v^2}{c^2}\right]^{1/2}$$

$$v = c\left[1 - \left(\frac{L}{L'}\right)^2\right]^{1/2} = c\left[1 - \left(\frac{75}{100}\right)^2\right]^{1/2} = 0.661c$$

1-10 (a) $\tau = \gamma\tau'$ where $\beta = \dfrac{v}{c}$ and

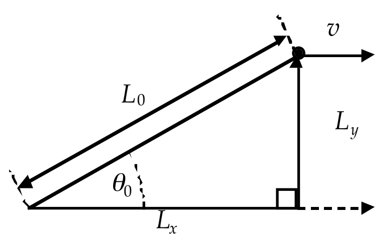
$$\gamma =' \left(1-\beta^2\right)^{-1/2} = \tau'\left(1 - \frac{v^2}{c^2}\right)^{-1/2} = \left(2.6\times 10^{-8}\ \text{s}\right)\left[1-(0.95)^2\right]^{-1/2} = 8.33\times 10^{-8}\ \text{s}$$

(b) $d = v\tau = (0.95)\left(3\times 10^{8}\right)\left(8.33\times 10^{8}\ \text{s}\right) = 24\ \text{m}$

1-12 (a) 70 beats/min or $\Delta t' = \dfrac{1}{70}$ min

(b) $\Delta t = \gamma\,\Delta t' = \left[1-(0.9)^2\right]^{-1/2}\left(\dfrac{1}{70}\right)$ min $= 0.032\,8$ min/beat or the number of beats per minute $\approx 30.5 \approx 31$.

1-14 (a) Only the x-component of L_0 contracts.



$$L_{x'} = L_0\cos\theta_0 \Rightarrow L_x = L_0\cos\theta_0/\gamma$$
$$L_{y'} = L_0\sin\theta_0 \Rightarrow L_y = L_0\sin\theta$$
$$L = \sqrt{L_x{}^2 + L_y{}^2} = \sqrt{L_0{}^2\cos^2\theta/\gamma^2 + L_0{}^2\sin^2\theta_0}$$
$$= L_0\sqrt{\cos^2\theta_0\left(1-\frac{v^2}{c^2}\right) + \sin^2\theta_0} = L_0\sqrt{1-\frac{v^2}{c^2}\cos^2\theta_0}$$

(b) As seen by the stationary observer, $\tan\theta = \dfrac{L_y}{L_x} = \dfrac{L_0\sin\theta_0}{L_0\cos\theta_0/\gamma} = \gamma\tan\theta_0$.

1-16 For an observer approaching a light source, $\lambda_{ob} = \left[\frac{(1-v/c)^{1/2}}{(1+v/c)^{1/2}}\right]\lambda_{source}$. Setting $\beta = \frac{v}{c}$ and after some algebra we find,

$$\beta = \frac{\lambda_{source}^2 - \lambda_{obs}^2}{\lambda_{source}^2 + \lambda_{obs}^2} = \frac{(650\text{ nm})^2 - (550\text{ nm})^2}{(650\text{ nm})^2 + (550\text{ nm})^2} = 0.166$$

$$v = 0.166c = (4.98\times 10^7\text{ m/s})(2.237\text{ mi/h})(\text{m/s})^{-1} = 1.11\times 10^8\text{ mi/h}.$$

1-17 (a) Galaxy A is approaching and as a consequence it exhibits blue shifted radiation. From Example 1.6, $\beta = \frac{v}{c} = \frac{\lambda_{source}^2 - \lambda_{obs}^2}{\lambda_{source}^2 + \lambda_{obs}^2}$ so that $\beta = \frac{(550\text{ nm})^2 - (450\text{ nm})^2}{(550\text{ nm})^2 + (450\text{ nm})^2} = 0.198$. Galaxy A is approaching at $v = 0.198c$.

(b) For a red shift, B is receding. $\beta = \frac{v}{c} = \frac{\lambda_{source}^2 - \lambda_{obs}^2}{\lambda_{source}^2 + \lambda_{obs}^2}$ so that $\beta = \frac{(700\text{ nm})^2 - (550\text{ nm})^2}{(700\text{ nm})^2 + (550\text{ nm})^2} = 0.237$. Galaxy B is receding at $v = 0.237c$.

1-18 (a) Let f_c be the frequency as seen by the car. Thus, $f_c = f_{source}\sqrt{\frac{c+v}{c-v}}$ and, if f is the frequency of the reflected wave, $f = f_c\sqrt{\frac{c+v}{c-v}}$. Combining these equations gives $f = f_{source}\frac{(c+v)}{(c-v)}$.

(b) Using the above result, $f(c-v) = f_{source}(c+v)$, which gives

$$(f - f_{source})c = (f + f_{source})v \approx 2f_{source}v.$$

The beat frequency is then $f_{beat} = f - f_{source} = \frac{2f_{source}v}{c} = \frac{2v}{\lambda}$.

(c) $$f_{beat} = \frac{2(30.0\text{ m/s})(10.0\times 10^9\text{ Hz})}{3.00\times 10^8\text{ m/s}} = \frac{2(30.0\text{ m/s})}{0.030\,0\text{ m}} = 2\,000\text{ Hz} = 2.00\text{ kHz}$$

$$\lambda = \frac{c}{f_{source}} = \frac{3.00\times 10^8\text{ m/s}}{10.0\times 10^9\text{ Hz}} = 3.00\text{ cm}$$

(d) $v = \frac{f_{beat}\lambda}{2}$ so,

$$\Delta v = \frac{\Delta f_{beat}\lambda}{2} = \frac{(5\text{ Hz})(0.030\,0\text{ m})}{2} = 0.075\,0\text{ m/s} \approx 0.2\text{ mi/h}$$