Department of Physics
University of California San Diego

Modern Physics (2D)
Prof. V. Sharma
Quiz # 8

## Some Relevant Formulae, Constants and Identities

$\lambda = \frac{h}{p}$ ; $\Delta x.\Delta p \geq \frac{h}{4\pi}$ ; $\Delta E.\Delta t \geq \frac{h}{4\pi}$

Time Dep. S. Eq: $-\frac{\hbar^2}{2m}\frac{d^2\Psi(x,t)}{dx^2} + U(x)\Psi(x,t) = i\hbar\frac{\partial\Psi(x,t)}{\partial t}$

Time Indep. S. Eq: $-\frac{\hbar^2}{2\mathrm{m}}\frac{d^2\psi(x)}{dx^2} + U(x)\psi(x) = E\ \psi(x)$

Particle in rigid box of length L: $\psi_n(0<x<L) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)$ & $\mathrm{E_n} = \frac{\mathrm{n}^2\pi^2\hbar^2}{2mL^2}$

Planck's constant h = $6.626\times10^{-34}$ J.s $=4.136\times10^{-15}$ eV.s , 1 eV = $1.60\times10^{-19}$ J

What to expect when expecting: $\langle Q\rangle = \int_{-\infty}^{\infty}\Psi^*(x,t)\,[\hat{Q}]\,\Psi(x,t)dx$

$[\hat{p}] = \frac{\hbar}{i}\frac{d}{dx}$ ; $[p^2] = -\hbar^2\frac{\partial^2}{\partial \mathrm{x}^2}$, $[\hat{\mathrm{K}}] = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial \mathrm{x}^2}$; $[\hat{\mathrm{E}}] = i\hbar\frac{\partial}{\partial \mathrm{t}}$

Uncertainty in Observable Q : $\Delta Q = \sqrt{<Q^2> - <Q>^2}$

$\sin\alpha\sin\beta = \frac{1}{2}\left(\cos(\alpha-\beta) - \cos(\alpha+\beta)\right)$

$\int u dv = uv - \int v du$

$\int \sin^2 x\, dx = = -\frac{1}{2}\cos x\sin x + \frac{1}{2}\mathrm{x} = \frac{1}{2}x - \frac{1}{4}\sin 2x$

$\int \cos^2 x\, dx = = \frac{1}{2}x + \frac{1}{4}\sin 2x$

$\int x\sin^2 x\, dx = \frac{\mathrm{x}^2}{4} - \frac{x\sin 2x}{4} - \frac{\cos 2x}{8}$

$\int x^2\sin^2 x\, dx = \frac{x^3}{6} - \left(\frac{x^2}{4} - \frac{1}{8}\right)\sin 2x - \frac{x\cos 2x}{4}$

Pl. write you answer in the Blue Book in indelible ink. Make sure your code number is prominently displayed on each page. Ask the proctor if you do not understand the question.

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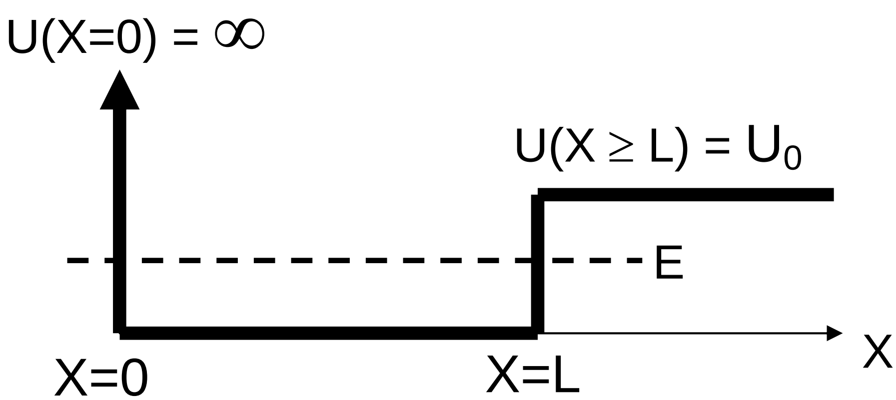
## Problem 1: Lazy "R" Us : Part II [12 pts]

Consider a particle of mass m moving in a one-dimensional box with rigid walls (infinite potential) at x = - L/2 and x = L/2. (a) Write down the expression for the normalized wavefunction for the ground state and sketch the wavefunction. Indicate the locations of the walls. Calculate (b) $\langle x \rangle$, (c) $\langle p \rangle$, (d) $\langle x^2 \rangle$, (e) $\langle p^2 \rangle$ and (f) calculate the product $\Delta x . \Delta p$ in this case and compare it with the expectation from the Uncertainty principle.
Note: $\sin^2 x + \cos^2 x = 1$

## Problem 2: Drawing Lesson ! [8 pts]

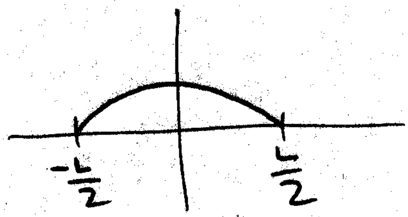
Consider a particle of mass m inside a square well having an infinite wall at x=0 and a wall of height $U_0$ at x=L. See figure below. For the case where the particle energy $E<U_0$ (a) write the solutions to Schrodinger's equation inside the well ( $0 \le x \le L$ ) and in the region beyond ($x > L$) that satisfy the appropriate boundary conditions at x=0 and x=L. (b) Separately sketch the probability density in all regions for the ground state and the first excited state of the particle.



# Phys 2D Quiz 8 Sols

1] a] Just like last week – translate $\sin\left(\frac{\pi x}{L}\right)$ by $\frac{L}{2}$ to get

$$\boxed{\psi(x) = \sqrt{\frac{2}{L}}\cos\left(\frac{\pi x}{L}\right)}$$




b] $\boxed{\langle x\rangle = 0}$ since the particle is bouncing back and forth. Or, do the integral:

$$\int_{-L/2}^{L/2} x\cos^2 x\,dx = 0$$

c] $\boxed{\langle p\rangle = 0,}$ also since the particle bounces back 'n' forth.

d] $$\langle x^2\rangle = \int_{-L/2}^{L/2}\left(\frac{2}{L}\right)x^2\cos^2\left(\frac{\pi x}{L}\right)dx$$

Use $\cos^2 x = 1-\sin^2 x \Rightarrow \frac{2}{L}\int_{-L/2}^{L/2} x^2\left(1-\sin^2\left(\frac{\pi x}{L}\right)\right)dx$

$$\int_{-L/2}^{L/2} x^2 = \frac{x^3}{3}\Big|_{L/2}^{L/2} = \frac{1}{3}\left(\frac{L}{2}\right)^3\cdot 2 = \frac{L^3}{12}$$

$$\int_{-L/2}^{L/2} x^2 \sin^2\left(\frac{\pi x}{L}\right)dx \;\; \underset{du = \frac{\pi}{L}dx}{\overset{u = \frac{\pi x}{L}}{=\!=\!=}} \;\; \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{L}{\pi}\right)^3 u^2 \sin^2 u \, du$$

$$= \left(\frac{L}{\pi}\right)^3 \left[\frac{u^3}{6} - \left(\frac{u^2}{4} - \frac{1}{8}\right)\sin 2u - \frac{u\cos 2u}{4}\right|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \left(\frac{L}{\pi}\right)^3 \left[\frac{1}{6}\left(\frac{\pi}{2}\right)^3 + \frac{\pi}{8} - \left(\frac{1}{6}\left(\frac{-\pi}{2}\right)^3 - \frac{\pi}{8}\right)\right]$$

$$= \left(\frac{L}{\pi}\right)^3 \cdot 2\left(\frac{\pi^3}{48} + \frac{\pi}{8}\right)$$

$$= L^3\left(\frac{1}{24} + \frac{1}{4\pi^2}\right)$$

So $\langle x^2 \rangle = \frac{2}{L}\left[\frac{L^3}{12} - L^3\left(\frac{1}{24} + \frac{1}{4\pi^2}\right)\right]$

$$= L^2\left[\frac{1}{6} - \frac{1}{12} - \frac{1}{2\pi^2}\right] = \boxed{L^2\left(\frac{1}{12} - \frac{1}{2\pi^2}\right)}$$

e] Use $E = \frac{p^2}{2m} \Rightarrow \langle p^2 \rangle = 2m\langle E \rangle = 2m\frac{\hbar^2\pi^2}{2mL^2} = \boxed{\frac{\hbar^2\pi^2}{L^2}}$

f] $\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \left(L^2\left(\frac{1}{12} - \frac{1}{2\pi^2}\right)\right)^{1/2}$

$\Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \left(\frac{\hbar^2\pi^2}{L^2}\right)^{1/2}$

$\Rightarrow \Delta x \Delta p = \left(\hbar^2\pi^2\left(\frac{1}{12} - \frac{1}{2\pi^2}\right)\right)^{1/2} = \left(\frac{\pi^2}{12} - \frac{1}{2}\right)^{1/2}\hbar \approx \boxed{0.56\hbar}$ bigger than $\frac{\hbar}{2}$!!

2] Sch. Eqn: $-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + U\psi = E\psi$

For $0 \le x \le L$, $U = 0$ $\Rightarrow$ $\frac{\partial^2\psi}{\partial x^2} = -\frac{2mE}{\hbar^2}\psi \equiv -k^2\psi$

So $\psi(x) = A\sin(kx) + B\cos(kx)$.

But since $\psi(0) = 0$ (b/c the potential goes to $\infty$ there)
we must have $B = 0$ $\Rightarrow$ $\psi = A\sin(kx)$ for $0 \le x \le L$

$x \ge L$: $U = U_0$ $\Rightarrow$ $\frac{\partial^2\psi}{\partial x^2} = \frac{2m(U_0 - E)}{\hbar^2}\psi \equiv \alpha^2\psi$

$\Rightarrow$ $\psi(x) = Ce^{\alpha x} + De^{-\alpha x}$.

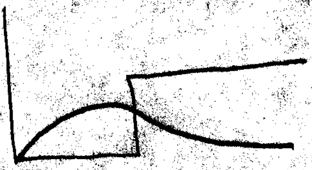
But $e^{\alpha x}$ blows up as $x \to \infty$, so $\psi(x) = De^{-\alpha x}$ for $x \ge L$.

Now, we need to enforce continuity & smoothness @ $x = L$

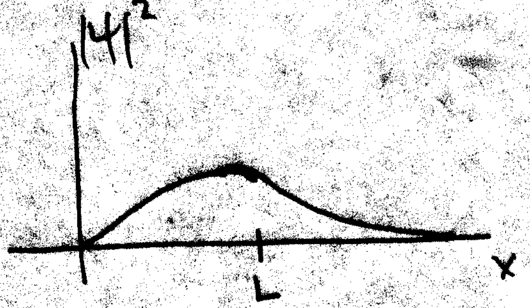
$$\boxed{\begin{array}{ll} \text{Continuous:} & A\sin(kL) = De^{-\alpha L} \\ \text{Smoothness:} & kA\cos(kL) = -\alpha De^{-\alpha L} \end{array}}$$

with

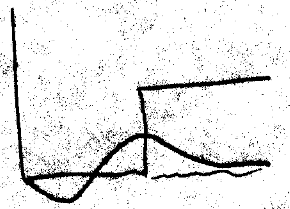
$$\boxed{\psi(x) = \begin{cases} A\sin(kx) & 0 \le x \le L \\ De^{-\alpha x} & x \ge L \\ (0 & x < 0) \end{cases}}$$

b) Groundstate wavefn is 

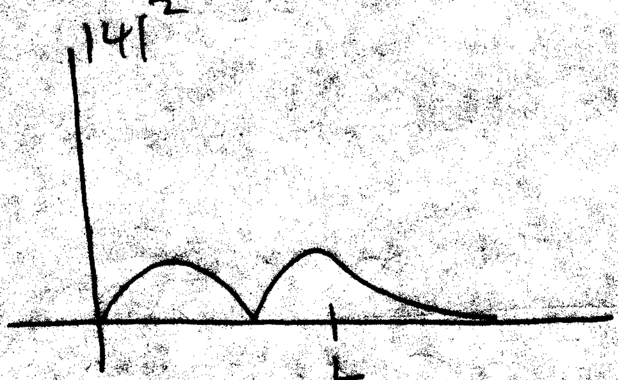
So the prob. density is just that squared:




(roughly)

First excited state (1 bump inside well): 

So prob. density is




(roughly).