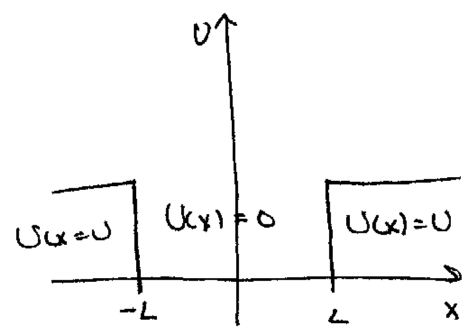
Physics 2D Homework - Ch. 5 Set 2

#20) Consider a particle with energy E bound to a finite square well of height U and width 2L situated on $-L \le x \le L$. Because the potential energy is symmetric about the midpoint of the well, the stationary state waves will be either symmetric (cosine solutions) or antisymmetric (sine solutions) about this point.

a.) Show that for $E < U$, the conditions for smooth joining of the interior and exterior waves lead to the following equation for the allowed energies of the symmetric waves.

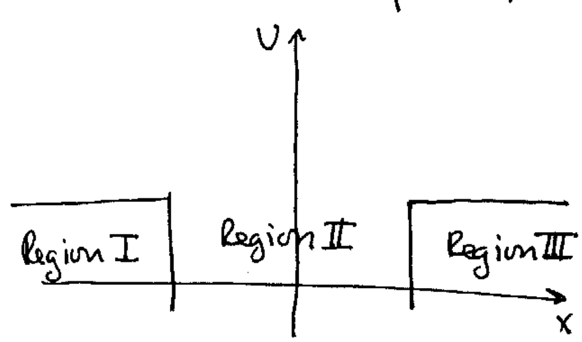
Let me start out by solving the finite square well problem for ya. Our box now looks like:



Since the potential is finite, there is some probability to find the particle in those regions. To find out how much, whip out the Schrödinger equation:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + U(x)\psi(x) = E\psi(x)$$

We need to divide up our problem into three regions:



In Region I, the potential $U(x)$ is $U(x)=U$, so the Schrödinger Eq. is:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}+U\psi=E\psi \qquad \underline{\text{Region I}}$$

In Region II, the potential is zero, so we get

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}=E\psi \qquad \underline{\text{Region II}}$$

Region III is the same as I:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}+U\psi=E\psi \qquad \text{Region III}$$

The solution to region II is easiest, cuz that's what you guys did last week:

$$\text{Region II}: \quad \frac{d^2\psi}{dx^2}+\frac{2mE}{\hbar^2}\psi=0=\frac{d^2\psi}{dx^2}+k^2\psi \qquad k^2=\frac{2mE}{\hbar^2}$$

$$\Rightarrow \psi = A\sin(kx)+B\cos(kx)$$

The solutions to Regions I and III are just a little more complicated

$$\text{Region I III}: \quad -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}+U\psi=E\psi$$

$$\frac{d^2\psi}{dx^2}+\frac{2m(E-U)}{\hbar^2}\psi=0$$

$$\frac{d^2\psi}{dx^2}+k'^2\psi=0 \qquad k'^2=\frac{2m(E-U)}{\hbar^2}$$

20.) continued

So far, it's exactly the same as Region II, except with the U thrown in. However, there's a little complication. We're told $E < U$, so that means k'^2 is negative. We can make it explicit by saying

$$k'^2 = \frac{2m(E-U)}{\hbar^2} = -\frac{2m(U-E)}{\hbar^2}$$

That means k' has to be imaginary!

$$k' = i\sqrt{\frac{2m(U-E)}{\hbar^2}} = i\alpha \quad ; \quad \alpha^2 = \frac{2m(U-E)}{\hbar^2}$$

So we have

$$\frac{d^2\psi}{dx^2} + k'^2\psi = 0$$

$$\Rightarrow \psi = Ce^{ik'x} + De^{-ik'x}$$

$$= Ce^{-\alpha x} + De^{\alpha x}$$ (could also be $C\sinh(\alpha x) + D\cosh(\alpha x)$ if you're into hyperbolic functions)

So let's list our solutions:

Region I: $\psi(x) = Ae^{\alpha x} + Be^{-\alpha x}$ $\qquad \alpha^2 = \frac{2m(U-E)}{\hbar^2}$

Region II: $\psi(x) = C\sin(kx) + D\cos(kx)$ $\qquad k^2 = \frac{2mE}{\hbar^2}$

Region III: $\psi(x) = Ee^{\alpha x} + Fe^{-\alpha x}$

We can simplify some stuff by demand ψ be finite.

In region I, as $x \to -\infty$, we want ψ to be finite (go to zero)

#20.) continued

So

Region I: $\psi(x \to -\infty) = \lim_{x \to -\infty} \left(Ae^{\alpha x} + Be^{-\alpha x} \right) = 0$

$$\lim_{x \to -\infty} e^{\alpha x} = 0$$

$$\lim_{x \to -\infty} e^{-\alpha x} = \infty$$

So to keep ψ finite,

$$B = 0$$

$\therefore$ In region I,

$$\psi(x) = Ae^{\alpha x}$$

Similarly, to keep region III finite as $x \to +\infty$,

Region III: $\psi(x \to \infty) = \lim_{x \to +\infty} \left(Ee^{\alpha x} + Fe^{-\alpha x} \right) = 0$

$$\lim_{x \to +\infty} e^{\alpha x} = \infty$$

$$\lim_{x \to +\infty} e^{-\alpha x} = 0$$

$$\therefore E = 0$$

So in Region III,

$$\psi(x) = Fe^{-\alpha x}$$

#20.) continued

Plus, in this problem, we're only worried about the symmetric (cosine) states. So now we're left with

Region I : $\psi(x) = Ae^{\alpha x}$

Region II : $\psi(x) = D\cos(kx)$

Region III : $\psi(x) = Fe^{-\alpha x}$

At the boundaries, we want to force ψ to be continuous, i.e.,

<u>at $x=-L$</u>

$$Ae^{-\alpha L} = D\cos(kL) \qquad (1)$$

<u>at $x=+L$</u>

$$Fe^{-\alpha L} = D\cos(kL) \qquad (2)$$

We also want ψ to be <u>smooth</u> at the boundries (smooth = $\frac{d\psi}{dx}$ is continuous)

<u>at $x=-L$</u>

$$\frac{d\psi}{dx}_{\text{Region I}} = \frac{d\psi}{dx}_{\text{Region II}}$$

$$\Rightarrow A\alpha e^{-\alpha L} = -Dk\sin(-kL) \qquad (3)$$

$$= Dk\sin(kL)$$

#20.) continued

at $x=L$

$$\frac{d\psi}{dx}_{\text{Region III}} = \frac{d\psi}{dx}_{\text{Region II}}$$

$$\Rightarrow -F\alpha e^{-\alpha L} = -D\sin(kL) \qquad (4)$$

Divide eq. (3) by (1) and (4) by (2) to get

$$\alpha = k\frac{\sin(kL)}{\cos(kL)}$$

$$\therefore \boxed{k\tan(kL) = \alpha}$$

b.) Show that the energy condition in a.) can be written as

$$k\sec(kL) = \sqrt{\frac{2mU}{\hbar^2}}$$

Since $\alpha^2 = \dfrac{2mU}{\hbar^2} - \dfrac{2mE}{\hbar^2}$

$$k^2 = \frac{2mE}{\hbar^2}$$

$$\alpha^2 + k^2 = \frac{2mU}{\hbar^2}$$

$$\therefore \alpha = \sqrt{\frac{2mU}{\hbar^2} - k^2}$$

#20.) continued

$$\therefore \quad k\tan(kL) = \sqrt{\frac{2mU}{\hbar^2} - k^2}$$

$$k^2\tan^2(kL) = \frac{2mU}{\hbar^2} - k^2$$

$$k^2\left(\tan^2(kL) + 1\right) = \frac{2mU}{\hbar^2}$$

Use the trig identity

$$\tan^2\theta + 1 = \sec^2\theta$$

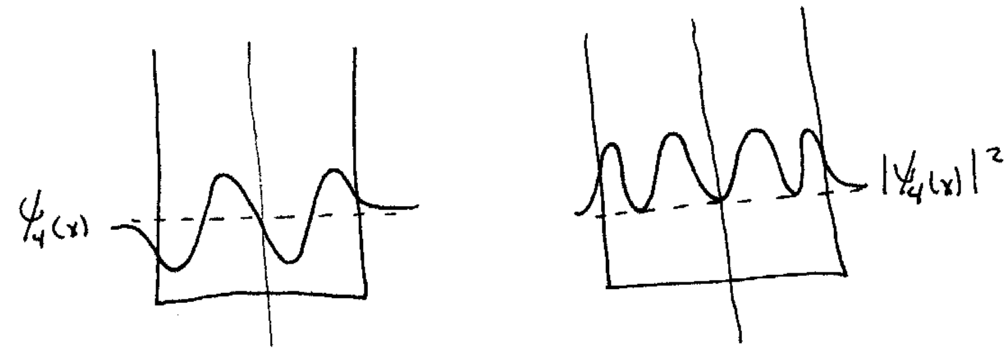
$$\therefore \quad \boxed{k\sec(kL) = \sqrt{\frac{2mU}{\hbar^2}}} \quad \blacksquare$$

Apply the result in this form to an electron trapped at a defect site in a crystal, modeling the defect as a square well of height 5 eV and width 0.2 nm. Write a simple computer program to find the ground state energy for the electron. Give your answer accurate to ± 0.001 eV.

You can do this using Excel like I did on that one problem way back when if you want, but it won't teach you anything other than how to work a computer. So I'll pass.

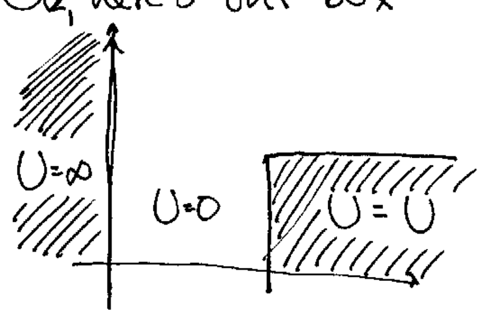
#21.) Sketch the wave function $\psi(x)$ and the probability density $|\psi(x)|^2$ for the n=4 state of a particle in a finite potential well

That will just be the next highest wave in figure 5.13, p. 202:



#23.) Consider a square well having an infinite wall at x=0 and a wall of height U at x=L. For the case of E<U, obtain solutions to the Schrödinger equation inside the well (0 ≤ x ≤ L) and in the region beyond (x>L) that satisfy the appropriate boundary conditions at x=0 and x=∞. Enforce the proper matching conditions at x=L to find the allowed energies of this system. Are there conditions for which no solution is possible? Explain.

Ok, here's our box



#25.) The wave function

$$\psi(x) = Cxe^{-\alpha x^2}$$

also describes a state of the quantum oscillator, provided the constant α is chosen properly.

a.) Using Schrödinger's equation, obtain an expression for α in terms of the oscillator mass m and the classical frequency of vibration ω. What is the energy of this state?

Schrödinger's equation for a quantum oscillator is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2}m\omega^2 x^2\psi = E\psi \qquad \text{(eq. 5.25)}$$

Take derivatives:

$$\frac{d^2\psi}{dx^2} = \frac{d^2}{dx^2}\left(Cxe^{-\alpha x^2}\right) = \frac{d}{dx}\left(Ce^{-\alpha x^2} - 2Cx^2\alpha e^{-\alpha x^2}\right)$$

$$= -2C\alpha xe^{-\alpha x^2} - 4Cx\alpha e^{-\alpha x^2} + 4Cx^3\alpha^2 e^{-\alpha x^2}$$

$$= 4C\alpha^2 x^3 e^{-\alpha x^2} - 6C\alpha xe^{-\alpha x^2}$$

So Erwin's equation up there looks like

$$-\frac{\hbar^2}{2m}\left(4C\alpha^2x^3 - 6C\alpha x\right)e^{-\alpha x^2} + \frac{1}{2}m\omega^2\left(Cx^3e^{-\alpha x^2}\right)$$

$$= ECxe^{-\alpha x^2}$$

#25.) continued

All those terms have C's and $e^{-\alpha x^2}$'s in 'em so cancel those out and rearrange to get

$$4\alpha^2 x^2 - 6\alpha = \left(\frac{m\omega}{\hbar}\right)^2 x^2 - \frac{2mE}{\hbar^2}$$

Equate coefficients of like powers of x:

$$4\alpha^2 = \left(\frac{m\omega}{\hbar}\right)^2$$

$$\Rightarrow \boxed{\alpha = \frac{m\omega}{2\hbar}}$$

and

$$6\alpha = \frac{2mE}{\hbar^2}$$

$$\therefore \boxed{E = \frac{3\alpha\hbar^2}{m} = \frac{3}{2}\hbar\omega}$$

b.) Normalize this wave

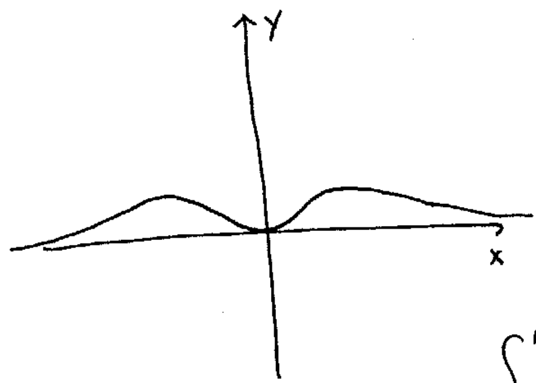
Okey dokey. The thing can be anywhere between $-\infty < x < \infty$, So

$$P = \int_{-\infty}^{\infty} |\psi(x)|^2 dx = 1$$

$$= C^2 \int_{-\infty}^{\infty} x^2 e^{-2\alpha x^2} dx$$

#25.) continued

To do this, we can use a keen little math trick. Note that the integrand is symmetric under $x \to -x$. If you plot it, it looks like:



See? It's symmetric about the y-axis. Now, an integral gives you the area under the curve, but since the area from $-\infty$ to zero is the same as from zero to ∞, we can write

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx$$
$$= \int_{0}^{\infty} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx$$

$$\therefore \int_{-\infty}^{\infty} f(x)\,dx = 2\int_{0}^{\infty} f(x)\,dx \quad \text{if} \quad f(x) = f(-x)$$

Sneaky! So now we can say

$$P = 2C^2 \int_0^{\infty} x^2 e^{-2\alpha x^2}\,dx$$

Still pretty hairy. The easiest thing to do is look it up, but you masochists out there can try to integrate by parts. Pain hurts me, though, so I'll look it up and say

$$P = 2C^2 \int_0^{\infty} x^2 e^{-2\alpha x^2}\,dx = 2C^2 \left(\frac{1}{8\alpha}\sqrt{\frac{\pi}{2\alpha}} \right) = 1$$

#25.) continued

So,

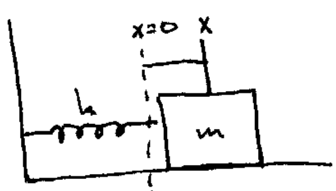
$$\boxed{C = \left(\frac{32\alpha^3}{\pi}\right)^{1/4}}$$

Sweet merciful god in heaven above!

#26.) Show that the oscillator energies in equation 5.29 correspond to the classical amplitudes

$$A_n = \sqrt{\frac{(2n+1)\hbar}{m\omega}}$$

Huh? Well, what they mean by that is that to solve the Quantum harmonic oscillator, we're using the potential $U = \frac{1}{2}m\omega^2x^2$. That's the same potential that you get when you do the ol' block on a spring problem! That problem looks like



The energy of this thing is its kinetic:

$$K = \frac{1}{2}mv^2$$

Plus its potential stored in the spring

$$U = \frac{1}{2}m\omega^2x^2$$

#29.) continued

$\langle x \rangle$ is the same classically and quantum mechanically.

$\langle x^2 \rangle$ differs by that factor of $\frac{L^2}{2\pi^2}$, but for general

n, you get

$$\langle x^2 \rangle = \frac{L^2}{3} - \frac{L^2}{2(n\pi)^2}$$

So for large n, we can drop it and recover the classical result.

#30.) An electron is described by the wave function

$$\psi(x) = \begin{cases} 0 & x < 0 \\ Ce^{-x}(1-e^{-x}) & x > 0 \end{cases}$$

where x is in nanometers and C is a constant.

a.) Find the value of C that normalizes ψ

$$P = \int_0^\infty C^2 e^{-2x}(1-e^{-x})^2 dx$$

$$= C^2 \int_0^\infty (e^{-2x} - 2e^{-3x} + e^{-4x}) dx$$

$$= C^2 \left[-\frac{e^{-2x}}{2} + \frac{2e^{-3x}}{3} - \frac{e^{-4x}}{4} \right] \Bigg|_0^\infty$$

$$= C^2 \left[\frac{1}{2} - \frac{2}{3} + \frac{1}{4} \right] = \frac{C^2}{12} = 1$$

$$\therefore \boxed{C = \sqrt{12}}$$

#30.) continued

b.) Where is the electron most likely to be found? That is, for what value of x is the probability of finding the electron largest?

The probability density is

$$|\psi(x)|^2 = 12(e^{-2x} - 2e^{-3x} + e^{-4x})$$

Maximize!

$$\frac{d|\psi(x)|^2}{dx} = 12(-2e^{-2x} + 6e^{-3x} - 4e^{-4x}) = 0$$

$$\Longrightarrow -1 + 3e^{-x} - 2e^{-2x} = 0$$

let $u = e^{-x}$

$$= 2u^2 - 3u + 1 = 0$$

$$\Rightarrow u = \frac{3 \pm \sqrt{9-8}}{4} = \frac{3 \pm 1}{4} = 1, \tfrac{1}{2}$$

$$u = e^{-x} = 1$$

$$\Rightarrow x = 0 \qquad \text{min}$$

$$u = e^{-x} = \tfrac{1}{2}$$

$$\Rightarrow \boxed{x = \ln 2 \qquad \text{max}}$$

#30.) continued

c.) calculate $\langle x \rangle$ for this electron and compare your results with its most likely position. Comment on any differences you find.

$$\langle x \rangle = \int_0^\infty \psi^* x \, \psi dx = C^2 \int_0^\infty x(e^{-2x} - 2e^{-3x} + e^{-4x}) \, dx$$

We need to know how to do an integral of the form

$$\int x e^{-ax} dx$$

When in doubt, there's no reason to be subtle. . Integrate it by parts :

$$\text{let } u = x \qquad dv = e^{-ax} dx$$
$$dx = du \qquad v = -\frac{e^{-ax}}{a}$$

$$\Rightarrow \int x e^{-ax} dx = -\frac{x e^{-ax}}{a} + \frac{1}{a} \int e^{-ax} dx$$

$$= -\frac{x e^{-ax}}{a} - \frac{e^{-ax}}{a^2}$$

$$\therefore \int_0^\infty x e^{-ax} dx = -\frac{x e^{-ax}}{a} \Big|_0^\infty - \frac{e^{-ax}}{a^2} \Big|_0^\infty$$

$$= \frac{1}{a^2}$$

#30.) continued

So,

$$\langle x \rangle = C^2\left(\frac{1}{4} - \frac{2}{9} + \frac{1}{16}\right) = \frac{13}{12}$$

$$\boxed{\langle x \rangle = 1.083\ nm}$$

#31.) For any eigenfunction ψ_n of the infinite squarewell, show that $\langle x \rangle = \frac{L}{2}$ and that

$$\langle x^2 \rangle = \frac{L^2}{3} - \frac{L^2}{2(n\pi)^2}$$

where L is the well dimension

For the infinite square well,

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$$

So

$$\langle x \rangle = \frac{2}{L}\int_0^L x \sin^2\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{2}{L}\int_0^L x \left[\frac{1}{2}\left(1 - \cos\left(\frac{2n\pi x}{L}\right)\right)\right] dx$$

$$= \frac{1}{L}\int_0^L \left[x - x\cos\left(\frac{2n\pi x}{L}\right)\right] dx$$

#32.) continued

Now we have to find the probability to locate the particle between $\langle x \rangle - \Delta x$ and $\langle x \rangle + \Delta x$:

$$P = \int_{\langle x \rangle - \Delta x}^{\langle x \rangle + \Delta x} |\psi(x)|^2 dx = \int_{-\frac{x_0}{\sqrt{2}}}^{\frac{x_0}{\sqrt{2}}} C^2 e^{-2|x|/x_0} dx$$

$$= 2C^2 \int_0^{\frac{x_0}{\sqrt{2}}} e^{-2\,\cdot\,/x_0} dx$$

$$= \frac{2C^2 e^{-2x/x_0}}{-2/x_0} \Bigg|_0^{\frac{x_0}{\sqrt{2}}}$$

$$\boxed{P = 1 - e^{-\sqrt{2}} = 0.757}$$

which is independent of x_0

#33.) Calculate $\langle x \rangle$, $\langle x^2 \rangle$, and Δx for a quantum oscillator in its ground state.

The ground state wavefunction of the quantum oscillator is:

$$\psi(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left[-\frac{m\omega}{2\hbar} x^2\right]$$

So

$$\langle x \rangle = \int_{-\infty}^{\infty} \psi^* x \psi dx = \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \int_{-\infty}^{\infty} x\, e^{-\frac{m\omega}{\hbar}x^2} dx$$

#3.) continued

Again, the integrand is odd, and the interval is symmetric

$\therefore \boxed{\langle x \rangle = 0}$

$$\langle x^2 \rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \int_{-\infty}^{\infty} x^2 e^{-\frac{m\omega}{\hbar}x^2} dx$$

Now the integrand is even, so

$$= 2\left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \int_{0}^{\infty} x^2 e^{-\frac{m\omega}{\hbar}x^2} dx$$

Now let's use their hint:

$$\int_0^\infty x^2 e^{-ax^2} dx = \frac{1}{4a}\sqrt{\frac{\pi}{a}}$$

with $a = \frac{m\omega}{\hbar}$

$$\boxed{\langle x^2 \rangle = \frac{\hbar}{2m\omega}}$$

$$\boxed{\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \left(\frac{\hbar}{2m\omega}\right)^{1/2}}$$

34 a) We expect $\langle p_x \rangle = 0$, since the particle is oscillating – just moves back and forth.

b) We know $E = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$.

$$\text{So } \langle E \rangle = \frac{\langle p^2 \rangle}{2m} + \frac{1}{2}m\omega^2 \langle x^2 \rangle$$

Since $\langle E \rangle = \frac{1}{2}\hbar\omega$ for the ground state,

$$\frac{1}{2}\hbar\omega = \frac{\langle p^2 \rangle}{2m} + \frac{1}{2}m\omega^2 \langle x^2 \rangle$$

$$\Rightarrow \langle p^2 \rangle = 2m\left[\frac{1}{2}\hbar\omega - \frac{1}{2}m\omega^2 \langle x^2 \rangle\right]$$

$$= m\hbar\omega - m^2\omega^2 \langle x^2 \rangle.$$

We know $\langle x^2 \rangle = \frac{\hbar}{2m\omega}$, so $\boxed{\langle p^2 \rangle = \frac{\hbar m\omega}{2}}$

c) $$\Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \boxed{\left(\frac{\hbar m\omega}{2}\right)^{1/2}}$$

Notice: $\Delta p \Delta x = \left(\frac{\hbar m\omega}{2}\right)^{1/2}\left(\frac{\hbar}{2m\omega}\right)^{1/2} = \frac{\hbar}{2}$ (The minimum uncertainty!)

#39.) Nonstationary states. Consider a particle in an infinite square well described initially by a wave that is a superposition of the ground and first excited states of the well

$$\Psi(x,0) = C[\psi_1(x) + \psi_2(x)]$$

a) Show that the value $C = \frac{1}{\sqrt{2}}$ normalizes this wave, assuming ψ_1 and ψ_2 are themselves normalized

First of all, you need to know that the $\psi_n(x)$'s of the square well are what's called orthogonal. That is,

$$\int_0^L \psi_n^*(x)\,\psi_m(x)\,dx = \begin{cases} 0 & \text{if } n \neq m \\ 1 & \text{if } n = m \end{cases}$$

You can show this using

$$\psi(x) = \sqrt{\frac{2}{L}}\,\sin\left(\frac{n\pi x}{L}\right)$$

But I'm not going to. Take my word for it. You might want to try proving it, though.

So let's normalize $\Psi(x,0)$

$$P = \int_0^L \Psi^*\,\Psi\,dx = \int_0^L C^2[\psi_1^*(x) + \psi_2^*(x)][\psi_1(x) + \psi_2(x)]\,dx$$

#39.) continued

$$= C^2 \int_0^L [\psi_1^* \psi_1 + \psi_2^* \psi_2 + \psi_1^* \psi_2 + \psi_2^* \psi_1]\, dx$$

Since the ψ_n's are orthogonal, the last two terms vanish

$$= C^2 \int_0^L [\psi_1^* \psi_1 + \psi_2^* \psi_2]\, dx$$

ψ_1 and ψ_2 are themselves normalized, so this is

$$= C^2 (1+1) = 2C^2 = 1$$

$$\therefore \boxed{C = \frac{1}{\sqrt{2}}}$$

b.) Find $\Psi(x,t)$ at any later time t.

Remember that to get the time dependent solution, you just multiply by

$$\phi_n(t) = e^{-i\frac{E_n}{\hbar}t}$$

$$\Psi(x,t) = C[\psi_1(x,t) + \psi_2(x,t)]$$

$$\boxed{\Psi(x,t) = C[\psi_1(x) e^{-\frac{iE_1}{\hbar}t} + \psi_2(x) e^{-\frac{iE_2}{\hbar}t}]}$$

#39.) continued

c.) Show that the superposition is not a stationary state, but that the average energy in this state is the arithmetic mean $(E_1+E_2)/2$ of the ground and first excited state energies E_1 and E_2.

What is meant by a stationary state is that the probability density does not change in time. So the way to show this is if something is a stationary state, then with respect to time the probability density is constant.

So

$$\frac{\partial|\psi(x,t)|^2}{\partial t}=0 \quad \text{if } \psi(x,t) \text{ is a stationary state}$$

Let's try it with this one

$$|\Psi(x,t)|^2 = C^2\left[\psi_1^*(x)\, e^{\frac{iE_1t}{\hbar}} + \psi_2^*\, e^{\frac{iE_2t}{\hbar}}\right]\left[\psi_1(x) e^{\frac{-iE_1t}{\hbar}} + \psi_2(x) e^{\frac{-iE_2t}{\hbar}}\right]$$

$$= C^2\Big[\psi_1^*(x)\,\psi_1(x) + \psi_2^*\,\psi_2(x) + \psi_1^*(x)\,\psi_2(x)\, e^{\frac{-i(E_2-E_1)t}{\hbar}}$$
$$+ \psi_2^*(x)\,\psi_1(x)\, e^{\frac{i(E_2-E_1)t}{\hbar}}\Big]$$

So

$$\boxed{\frac{\partial|\Psi(x,t)|^2}{\partial t} \neq 0!}$$

Not a stationary state!

#39.) continued

To get the average energy

$$\langle E \rangle = \int_0^L \Psi^*_{(x,t)}\, \hat{E}\, \Psi_{(x,t)}\, dx$$

Remember that

$$\hat{E} = i\hbar \frac{\partial}{\partial t}$$

$$= C^2 \int_0^L \left[\psi_1^*(x) e^{\frac{iE_1 t}{\hbar}} + \psi_2^*(x) e^{\frac{iE_2 t}{\hbar}}\right] \left(i\hbar \frac{\partial}{\partial t}\right) \left[\psi_1(x) e^{-\frac{iE_1 t}{\hbar}} + \psi_2(x) e^{-\frac{iE_2 t}{\hbar}}\right] dx$$

$$= i\hbar C^2 \int_0^L \left[\psi_1^* e^{\frac{iE_1 t}{\hbar}} + \psi_2^* e^{\frac{iE_2 t}{\hbar}}\right]\left[-i\frac{E_1}{\hbar}\psi_1(x) e^{-\frac{iE_1 t}{\hbar}} - i\frac{E_2}{\hbar}\psi_2(x) e^{-\frac{iE_2 t}{\hbar}}\right] dx$$

$$= C^2 \int_0^L \left[E_1 \psi_1^* \psi_1 + E_2 \psi_2^* \psi_2\right] dx$$

All the other terms go away thanks to orthogonality!

$$= C^2 (E_1 + E_2)$$

$$= \boxed{\frac{(E_1 + E_2)}{2} = \langle E \rangle}$$

Oh my god! Time for a Bud!

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$$\langle x \rangle = \int \Psi^*(x,t)\, x\, \Psi(x,t)$$

$$= \int \left(\frac{1}{\sqrt{2}}\right)\left(\psi_1^*(x)e^{+i\omega_1 t} + \psi_2^*(x)e^{+i\omega_2 t}\right) x \left(\frac{1}{\sqrt{2}}\right)\left(\psi_1(x)e^{-i\omega_1 t} + \psi_2(x)e^{-i\omega_2 t}\right)$$

$$= \frac{1}{2}\left(\int x|\psi_1|^2 dx + \int x|\psi_2|^2 dx\right) + \frac{1}{2}\int\left(\psi_1\psi_2^* e^{i(\omega_2-\omega_1)t} x + \psi_2\psi_1^* e^{-i(\omega_1-\omega_2)t} x\right)dx$$

Now, since ψ_1 & ψ_2 are real, $\psi_1\psi_2^* = \psi_2\psi_1^*$

and $e^{i\theta} + e^{-i\theta} = 2\cos\theta$, so

$$\frac{1}{2}\int\left(\psi_1\psi_2^* e^{i(\omega_2-\omega_1)t} x + \psi_2\psi_1^* e^{-i(\omega_1-\omega_2)t} x\right)dx = \frac{1}{2}\int x\psi_1^*\psi_2 \cdot 2\cos((\omega_2-\omega_1)t)\,dx$$

$$= \int x\psi_1^*\psi_2 \cos\left(\frac{E_2-E_1}{\hbar}t\right)$$

Done! $\langle x \rangle = \left[\frac{1}{2}\left(\int x|\psi_1|^2 dx + \int x|\psi_2|^2 dx\right)\right] + \left[\int x\psi_1^*\psi_2\, dx\right]\cos\left(\frac{E_2-E_1}{\hbar}t\right)$

For $L = 1\,nm$,

$$X_0 = \frac{1}{2}\left[\langle x\rangle_1 + \langle x\rangle_2\right] = \frac{1}{2}\left[\frac{L}{2} + \frac{L}{2}\right] = \boxed{\frac{L}{2}}$$

and $$A = \int x\,\psi_1^*\psi_2 = \frac{2}{L}\int_0^L x \sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{2\pi x}{L}\right)dx$$

$$= \frac{1}{L}\int_0^L x\left[\cos\left(\frac{\pi x}{L}\right) - \cos\left(\frac{3\pi x}{L}\right)\right]dx$$ (I used a trig identity).

Now integrate by parts:

$$\int x\cos(ax)\,dx = \frac{1}{a}x\sin(ax) - \int \frac{1}{a}\sin(ax)$$

$$\Rightarrow A = \frac{1}{L}\left[\frac{L}{\pi}\int_0^L \sin\left(\frac{\pi x}{L}\right) + \frac{L}{3\pi}\int_0^L \sin\left(\frac{3\pi x}{L}\right)\right]$$

$$= +\left(\frac{L}{\pi}\right)^2(-2) - \left(\frac{L}{3\pi}\right)^2(-2)$$

$$= L^2\left(\frac{2}{9\pi^2} - \frac{2}{\pi^2}\right) = \boxed{\frac{-16L}{9\pi^2}} = -0.18\,nm \text{ for } L = 1\,nm.$$

Time to shuttle back 'n' forth = Period = $\frac{1}{F} = \frac{2\pi}{\omega}$

Here, $\omega = \frac{E_2 - E_1}{\hbar} = \frac{3E_1}{\hbar} = \frac{3\pi^2\hbar}{2mL^2}$

So $\frac{2\pi}{\omega} = \frac{4mL^2}{3\pi\hbar} = \boxed{3.67\times10^{-15}\,s}$

Clasically, electron would have $v = \left(\frac{2E}{m}\right)^{1/2} = \left(\frac{E_1+E_2}{m}\right)^{1/2}$

$E_1 + E_2 = \frac{5\pi^2\hbar^2}{2mL^2}$, so $v = \left(\frac{5\pi^2\hbar^2}{2m^2L^2}\right)^{1/2} = \sqrt{\frac{5}{2}}\,\frac{\pi\hbar}{mL} = \boxed{5.75\times10^{5}\,\frac{m}{s}}$

So it would take time $2\left(\frac{L}{v}\right) = \boxed{3.47\times10^{-15}\,s}$ clasically!