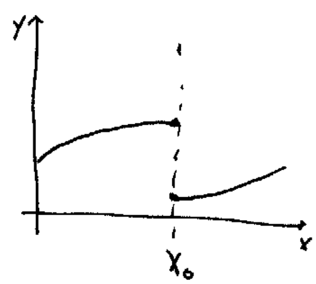
Physics 2D Homework Solutions - Ch. 5

#1.) Of the functions graphed in Figure P5.1, which are candidates for the Schrödinger wavefunction of an actual physical system? For those that are not, state why they fail to qualify.

There's only a few absolute requirements for a wavefunction:

1.) Ψ represents a probability for locating a particle at a point, and should therefore be <u>single-valued</u> for any given x. It doesn't make any sense for there to be different values of the probability located at the same point.

2.) This implies that Ψ should also be <u>continuous</u>; no big ugly jumps. Something discontinuous, like



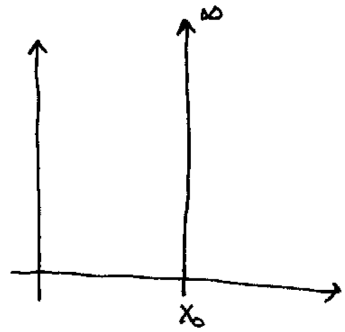
Is double valued at x_0, if we examine

$$\lim_{x \to x_0^+} \psi \neq \lim_{x \to x_0^-} \psi$$

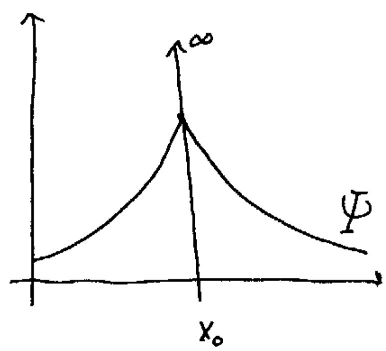
3.) Most of the time, we will also require Ψ to be <u>smooth</u>, that is, require $\frac{d\Psi}{dx}$ to be continuous.

#1.) continued

This can be violated, though, for example near infinite potentials. For example, something called a "delta function potential" looks like a single infinite spike at some point x_0



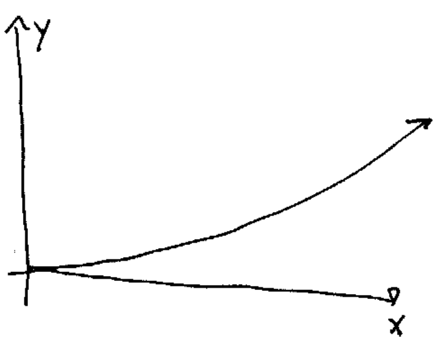
At x_0, Ψ won't be smooth, but it will be continuous and single valued:



4.) Last, we require Ψ to be finite for <u>all</u> values of x (even $x \to \pm\infty$). No infinite probabilities!

Down to business

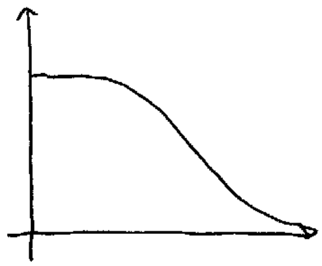
a.)



No good! As $x \to \infty$ $\Psi \to \infty$, so it violates Kuhlman golden rule #4.

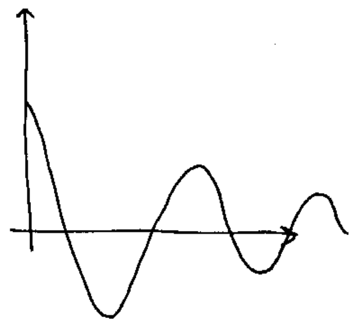
#1.) continued

b.)



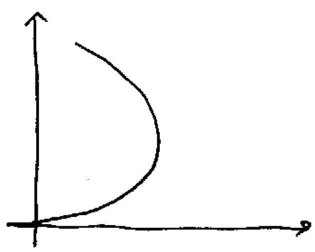
Nice, smooth, finite, and continuous.
Looks good!

c.)



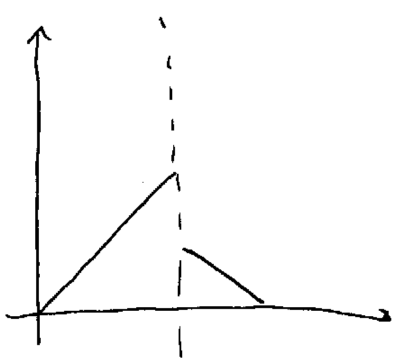
Even looks wavy! What more could you ask for?

d.)



Stinky! Double valued for some x.
No go.

e.)



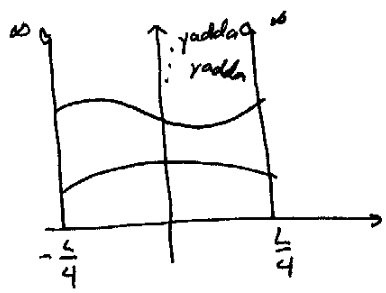
Even worse! At least d.) was continuous
Letter e.), you suck.

#2.) A particle is described by the wavefunction

$$\psi(x) = \begin{cases} A\cos\left(\frac{2\pi x}{L}\right) & -\frac{L}{4} \le x \le \frac{L}{4} \\ 0 & \text{otherwise} \end{cases}$$

a.) Determine the normalization constant A.

First of all, this is our old friend particle in a box, where the box is length $\frac{L}{2}$



So we could just cheat and say "We found that in section 5.4. Let's use that and let $L \to \frac{L}{2}$!"

This way is for wussies! Let's tough it out so we can look ourselves in the mirror.

$|\psi(x)|^2$ is the probability amplitude, and to find the probability to find a particle in the regim $a \le x \le b$,

$$P = \int_a^b |\psi(x)|^2 dx$$

Now, our particle is inside this box, so we know

$$P = \int_{-\frac{L}{4}}^{\frac{L}{4}} |\psi(x)|^2 dx$$

#2.) continued

It HAS to be in the box somewhere! So the probability to locate it in there is 100%

$$P = \int_{-\frac{L}{4}}^{\frac{L}{4}} |\psi(x)|^2 dx = 1$$

$$= \int_{-\frac{L}{4}}^{\frac{L}{4}} A^2 \cos^2\left(\frac{2\pi x}{L}\right) dx$$

$$= A^2 \int_{-\frac{L}{4}}^{\frac{L}{4}} \cos^2\left(\frac{2\pi x}{L}\right) dx$$

Let $u = \frac{2\pi x}{L}$ when $x = -\frac{L}{4}$, $u = -\frac{\pi}{2}$

$du = \frac{2\pi}{L} dx$ $x = \frac{L}{4}$, $u = \frac{\pi}{2}$

$\Rightarrow dx = \frac{L}{2\pi} du$

$$= \frac{A^2 L}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 u \, du$$

Remember how to do this one?

$$= \frac{A^2 L}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}(1 + \cos 2u) \, du$$

$$= \frac{A^2 L}{2\pi} \left[\frac{u}{2} + \frac{\sin 2u}{4} \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{A^2 L}{4}$$

#2.) continued

And so we know

$$\frac{A^2 L}{4} = 1$$

$$\therefore \boxed{A = \sqrt{\frac{4}{L}}}$$

b.) What is the probability that the particle will be found between $x=0$ and $x=\frac{L}{8}$ if a measurement of its position is made?

Just do

$$P = \int_0^{\frac{L}{8}} |\psi(x)|^2 dx$$

$$= \frac{4}{L}\int_0^{\frac{L}{8}} \cos^2\left(\frac{2\pi x}{L}\right) dx$$

Do this the same way as part a.)

$$= \frac{2}{\pi}\int_0^{\frac{\pi}{4}} \cos^2 u \, du$$

$$= \frac{2}{\pi}\left[\frac{u}{2} + \frac{\sin 2u}{4}\right]\Bigg|_0^{\frac{\pi}{4}}$$

$$\boxed{P = \frac{2}{\pi}\left(\frac{\pi}{8} + \frac{1}{4}\right)}$$

#3.) A free electron has a wavefunction

$$\psi(x) = A\sin(5\times10^{10}x)$$

where x is measured in meters (That means 5×10^{10} has units m^{-1}!) Find

a.) the electron's de Broglie wavelength

For a free particle, the wavefunction <u>always</u> has the form

$$\psi(x) = A\sin(kx+\phi)$$

So we know that in our case, $\phi = 0$ and

$$k = 5\times10^{10} \equiv \frac{2\pi}{\lambda}$$

$$\therefore \boxed{\lambda = \frac{2\pi}{5\times10^{10}} = 0.126\ \text{nm}}$$

b.) the electron's momentum

Well, you should remember

$$\lambda = \frac{h}{p} \Rightarrow \boxed{p = \frac{h}{\lambda} = 5.26\times10^{-24}\ \frac{kg\,m}{s} = 32.8\ \mu\frac{eV}{c}}$$

That's a little sissy momentum. Using $p = mv$ gives you $v = 0.01\ c$, so we're non relativistic

#3.) continued

c.) the electron's energy in electron volts

$$\boxed{K = \frac{p^2}{2m} = 15.1 \times 10^{-18}\ \text{J} = 94\ \text{eV}}$$

#5.) In a region of space, a particle with zero energy has a wave function

$$\psi(x) = A x e^{-x^2/L^2}$$

a.) Find the potential energy U as a function of x.

The Schrödinger equation has the form

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + U(x)\psi(x) = E\psi(x)$$

For our particle, E=0, so

$$\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} = U(x)\psi(x)$$

$$\Rightarrow U(x) = \frac{1}{\psi(x)}\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2}$$

Right? Plug in our wavefunction.

#5.) continued

$$U(x) = \frac{1}{Axe^{-x^2/L^2}} \frac{\hbar^2}{2m} \frac{d^2}{dx^2}\left(Axe^{-x^2/L^2}\right)$$

Take Those derivatives! (Don't forget chain + product rule!)

$$\frac{d^2}{dx^2}\left(Axe^{-x^2/L^2}\right) = \frac{d}{dx}\left(Ae^{-x^2/L^2} - \frac{2A}{L^2}x^2e^{-x^2/L^2}\right)$$

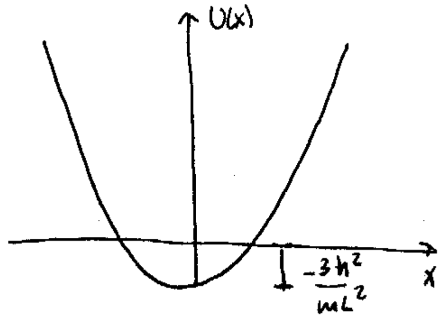
$$= -\frac{2Ax}{L^2}e^{-x^2/L^2} - \frac{4Ax}{L^2}e^{-x^2/L^2} + \frac{4Ax^3}{L^4}e^{-x^2/L^2}$$

$$= \frac{2Ax(2x^2-3L^2)e^{-x^2/L^2}}{L^4}$$

So $$\boxed{U(x) = \frac{\hbar^2}{2mL^2}\left(\frac{4}{L^2}x^2 - 6\right)}$$

b.) Make a sketch of U(x) versus x

Parabola centered at $x=0$ with $U(0) = -\frac{3\hbar^2}{mL^2}$



#6.) continued

So that

$$\boxed{\hat{H}\psi_{(x)} = \frac{\hbar^2 k^2}{2m}\cdot\psi_{(x)}}$$

(note that this is $\frac{p^2}{2m}\psi$ for $p=\hbar k$)

That constant multiplying $\psi_{(x)}$ is the energy

$$\hat{H}\psi_{(x)} = E\,\psi_{(x)}$$

$$\therefore \boxed{E = \frac{\hbar^2 k^2}{2m} = \frac{p^2}{2m}}$$

#7) Show that allowing the state $n=0$ for a particle in a one dimensional box violates the uncertainty principle, $\Delta x\cdot\Delta p \geq \frac{\hbar}{2}$

The particle is somewhere inside the box, right?

So $\Delta x = L$ (L = box length)

For a particle in a box,

$$E_n = \frac{n^2 \pi^2 \hbar^2}{2mL^2} \qquad \text{(Eq. 5.17)}$$

for $n=0$

$$E_n = 0$$

#7.) Inside the box, all energy is kinetic

$$E = K = \frac{\langle p^2 \rangle}{2m} = 0$$

$$\therefore \langle p^2 \rangle = 0$$

Now,

$$\psi(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$$

So that

$$\langle p \rangle = \int_0^L \psi^*(x)\, \hat{p}\, \psi(x)\, dx$$

$$= \frac{2}{L} \int_0^L \sin\left(\frac{n\pi x}{L}\right) \left(-i\hbar \frac{d}{dx}\right) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$= -i\hbar \left(\frac{2n\pi}{L^2}\right) \int_0^L \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx$$

$$= 0$$

So that

$$\Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} \qquad \text{Eq. 5.34}$$

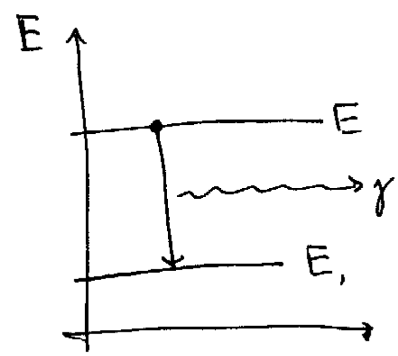
$$= 0$$

$$\therefore \boxed{\Delta x \, \Delta p = 0}$$

#9.) The nuclear potential that binds protons and neutrons in the nucleus of an atom is often approximated by a square well. Imagine a proton confined in an infinite square well of length 10^{-5} nm, a typical nuclear diameter. Calculate the wavelength and energy associated with the photon that is emitted when the proton undergoes a transition from the first excited state ($n=2$) to the ground state ($n=1$). In what region of the electromagnetic spectrum does this wavelength belong?

This is easier than it sounds. We are describing the situation as a particle in a box, so we know the energy states are

$$E_n = \frac{n^2 \pi^2 \hbar^2}{2 m_p L^2}$$



So $$E_2 = \frac{4 \pi^2 \hbar^2}{2 m_p L^2}$$

$$E_1 = \frac{\pi^2 \hbar^2}{2 m_p L^2}$$

The emitted photon has the difference in energy between these two states

#9.) continued

$$\boxed{\Delta E = \frac{3\pi^2\hbar^2}{2m_p L^2} = 6.14 \text{ MeV}}$$

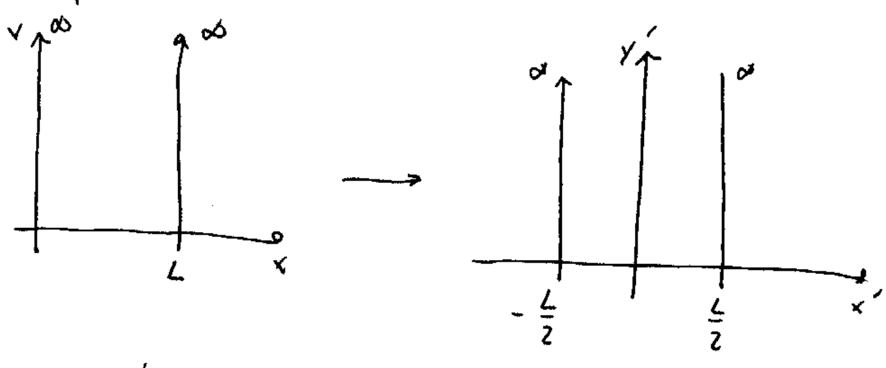
$$\Delta E = \frac{hc}{\lambda} \Rightarrow \boxed{\lambda = \frac{hc}{\Delta E} = 2.02 \times 10^{-4} \text{nm}}$$

That's in the gamma ray region

#11.) Consider a particle moving in a one dimensional box with walls at $x = -\frac{L}{2}$ and $x = \frac{L}{2}$.

a.) Write the wave functions and probability densities for the states $n=1$, $n=2$, and $n=3$

This is just like the box with ends at $x=0$ and $x=L$, except shifted to the left:



It's the exact same problem, so the wavefunctions should be the same, except with

$$x = x' + \frac{L}{2}$$