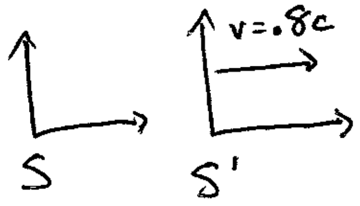
2D Final Exam Solns

1] We need to figure out the messenger's speed in the armada's frame. Let S' be the armada frame.



So $u' = \dfrac{u-v}{1-\frac{uv}{c^2}} = 0.625c$

a] In the armada frame, the messenger travels a distance of 1 ly moving at speed 0.625c.

So time $= \dfrac{1\ ly}{0.625c} = \boxed{1.6\,yr = 5.0\times10^7\,sec}$

b] In the messenger's frame, the armada is

$\dfrac{1\ ly}{\gamma} = 1\,ly\sqrt{1-(.625)^2} = 0.78\,ly$ long and moving at a speed 0.625c. So it takes $\dfrac{0.78\,ly}{.625c} = \boxed{1.25\,yr}$ $= \boxed{3.9\times10^7\,sec}$

c] Now we need to be careful — since the events do <u>not</u> take place at the same point in any frame but the messenger's, we can't just naively time dilate.

Time dilating the messenger's time using $u = 0.95c$

gives $\dfrac{1.25\text{ yr}}{\sqrt{1-.95^2}} = \boxed{4\text{ yr} = 1.26\times10^{8}\text{ sec}}$

(Note: We could also get this in the Earth frame w/o doing a time dilation. In this frame, the armada is $(1\text{ ly})(\sqrt{1-.8^2}) = 0.6\text{ ly}$ long. Say it takes time t for the messenger to complete the trip – that means he'll have moved a distance $(0.95c)t$.

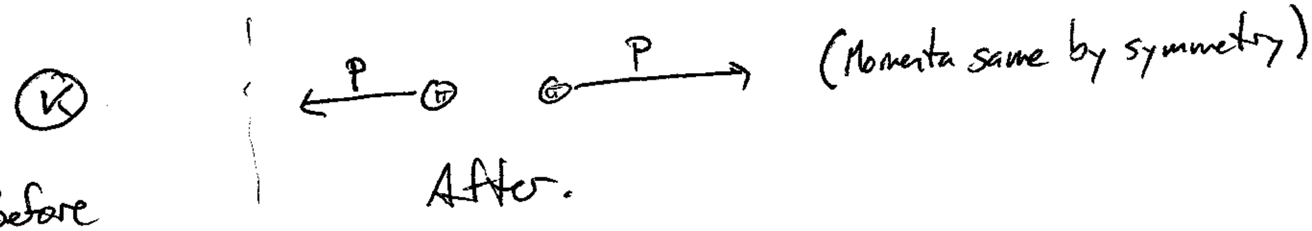
But that's the same as the length of the armada plus the distance it moved, which is $0.6\text{ ly} + (0.8c)t$

$$\Rightarrow \quad 0.6\text{ ly} + .8ct = .95ct$$

$$\Rightarrow \quad 0.6\text{ ly} = .15ct$$

$$\Rightarrow \quad t = \frac{0.6}{.15}\text{ yr} = 4\text{ yr}.$$ Same as before!

2] In the K^0 frame, it starts @ rest and then decays



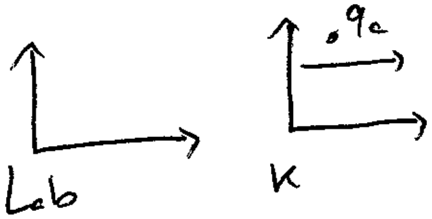
Conservation of energy tells us $m_K c^2 = 2\gamma m_\pi c^2$

So $\gamma = \dfrac{m_K}{2m_\pi} = 1.78$

$$\Rightarrow \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = 1.78 \Rightarrow 1 - \frac{v^2}{c^2} = \frac{1}{(1.78)^2}$$

$$\Rightarrow v = \sqrt{1 - \frac{1}{(1.78)^2}}\, c = 0.827c$$

In the lab frame, we need to do a velocity transformation.



The pion moving to right is the ~~[illegible]~~ fastest:

a] $$u_{lab} = \frac{u_K + v}{1 + \frac{u_K v}{c^2}} = \frac{.827c + .9c}{1 + (.827)(.9)} = \boxed{0.99c}$$

b] The one moving to the left is slower:

$$u_{lab} = \frac{-.827c + .9c}{1 - .827(.9)} = \boxed{0.28c}$$

3]

a] $KE_{max} = \frac{hc}{\lambda} - \phi = \frac{1240 \text{ eV}\cdot\text{nm}}{200 \text{ nm}} - 4.2 \text{ eV}$

$$= \boxed{2 \text{ eV}}$$

b] $eV_s = KE_{max} \Rightarrow \boxed{V_s = 2V}$

c] $\frac{hc}{\lambda} = \phi \Rightarrow \lambda = \frac{1240 \text{ eV}\cdot\text{nm}}{4.2 \text{ eV}} = \boxed{295 \text{ nm}}$

d] Energy of 1 photon is $\frac{hc}{\lambda} = 6.2 \text{ eV} = 9.92\times10^{-19} \text{ J}$

So If $2 \frac{W}{m^2} = \frac{2J}{sec\cdot m^2}$ are incident, that's

$$\frac{2J}{9.92\times10^{-19} J} = \boxed{2\times10^{18} \text{ photons}/\text{sec}\cdot\text{m}^2}$$

4]

a] We need to use the reduced mass here, since the "nucleus" is of a comparable ~~size~~ mass to the electron's

$$\mu = \frac{m_{e^+} m_{e^-}}{m_{e^+} + m_{e^-}} = \frac{m_e^2}{2m_e} = \frac{m_e}{2}. \quad (\text{since } m_{e^+} = m_{e^-}).$$

So now just replace m_e with $\frac{m_e}{2}$ in the Bohr formulae:

$$r_1 = \frac{\hbar^2}{\mu k e^2} = \frac{2\hbar^2}{m_e k e^2} = \boxed{2a_0} = \boxed{1.\ \text{Å}}$$

b] $$E_1 = \frac{-ke^2}{2\left(\frac{\hbar^2}{\mu k e^2}\right)} = -\mu \frac{k^2 e^4}{2\hbar^2} = -\frac{1}{2}\left(\frac{m_e k^2 e^4}{2\hbar^2}\right) = \boxed{-\frac{1}{2}(13.6\,\text{eV})}$$

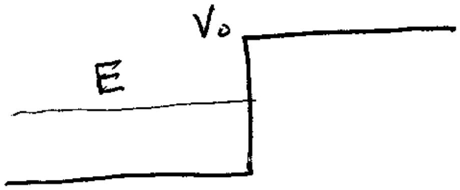
So $\boxed{E_1 = -6.8\,\text{eV}}$

c] E_n for positronium $= \frac{1}{2} E_n$ for hydrogen.

So $$\lambda = \frac{hc}{\Delta E_{pos}} = \frac{hc}{\frac{\Delta E_{hydrogen}}{2}} = 2\left(\frac{hc}{\Delta E_H}\right)$$

The wavelengths are $\boxed{2 \text{ times}}$ what they are for hydrogen.

5]



a] Sch. Eqn: $-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} = E\psi \Rightarrow \frac{\partial^2\psi}{\partial x^2} = -\frac{2mE}{\hbar^2}\psi$

$$\Rightarrow \boxed{\psi = Ae^{ikx} + Be^{-ikx}, \quad k = \sqrt{\frac{2mE}{\hbar^2}}}$$

(we can't get rid of either component, since A is the incident beam and B is reflected).

b] Sch. Eqn: $-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V_0\psi = E\psi$

$$\Rightarrow \frac{\partial^2\psi}{\partial x^2} = \frac{2m(V_0 - E)}{\hbar^2}\psi$$

$$\Rightarrow \psi = Ce^{-\alpha x} + De^{\alpha x}, \quad \alpha = \sqrt{\frac{2m(V_0-E)}{\hbar^2}}$$

But we can lose D 'cause it blows up as $x \to \infty$.

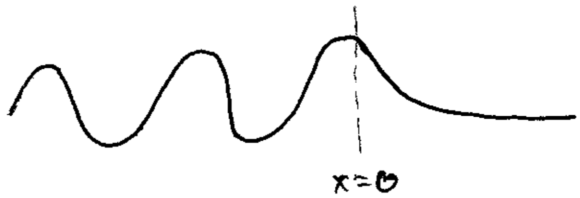
$$\Rightarrow \boxed{\psi = Ce^{-\alpha x}, \quad \alpha = \sqrt{\frac{2m(V_0-E)}{\hbar^2}}}$$

c] Continuous: $\psi_{x<0} = \psi_{x>0}$ at $x=0$

$$\Rightarrow \boxed{A+B=C}$$

Smoove: Derivatives match: $\boxed{ikA - ikB = -\alpha C}$

d] Oscillates then decays:



e] $P = |\psi|^2$. Inside the barrier, we get

$P = C^2 e^{-2\alpha x}$. At $\boxed{x = \frac{1}{2\alpha}}$ this will be $\frac{1}{e}C^2 = \frac{P(x=0)}{e}$.

6]

a] $$\boxed{-\frac{\hbar^2}{2m}\nabla^2\psi(x,y,z) + \frac{1}{2}m(\omega_1^2x^2+\omega_2^2y^2+\omega_3^2z^2)\psi(x,y,z) = E\psi(x,y,z)}$$

b] We can write $\psi(x,y,z) = X(x)Y(y)Z(z)$.

But first we need to figure out the quantum numbers of the 1^{st} excited state.

$E = (n_x + \frac{1}{2})\hbar\omega_1 + (n_y + \frac{1}{2})\hbar\omega_2 + (n_z + \frac{1}{2})\hbar\omega_3$

Ground state is $n_x = n_y = n_z = 0$.

Since $\omega_1 < \omega_2 < \omega_3$, the 1st excited state is $n_x = 1$, $n_y = n_z = 0$.

So $X(x) = A_1 x e^{-\frac{m\omega_1}{2\hbar}x^2}$ ($n_x = 1$, so 1st excited state)

$Y(y) = A_2 e^{-\frac{m\omega_2}{2\hbar}y^2}$

$Z(z) = A_3 e^{-\frac{m\omega_3}{2\hbar}z^2}$

both ground states.

where A_1, A_2, A_3 are normalization constants.

$$\therefore \boxed{\psi(x,y,z) = A_1 A_2 A_3\, x \cdot \exp\left[-\frac{m\omega_1}{2\hbar}x^2 - \frac{m\omega_2}{2\hbar}y^2 - \frac{m\omega_3}{2\hbar}z^2\right]}$$

c] $\int_{-\infty}^{\infty} |\psi|^2 = 1$. Let's normalize each part separately.

$$\int_{-\infty}^{\infty} A_1^2 x^2 e^{-\frac{m\omega_1}{\hbar}x^2} = A_1^2 \frac{\pi^{1/2}}{2\left(\frac{m\omega_1}{\hbar}\right)^{3/2}} \Rightarrow A_1 = \frac{2^{1/2}\, m^{3/4}\omega_1^{3/4}}{\pi^{1/4}\hbar^{3/4}}$$

$$\int_{-\infty}^{\infty} A_2^2 e^{-\frac{m\omega_2}{\hbar}y^2} = A_2^2 \frac{\pi^{1/2}\hbar^{1/2}}{m^{1/2}\omega_2^{1/2}} \Rightarrow A_2 = \left(\frac{m\omega_2}{\pi\hbar}\right)^{1/4}$$

$$\int_{-\infty}^{\infty} A_3^2 e^{-\frac{m\omega_3}{\hbar}z^2} = A_3^2 \left(\frac{\pi\hbar}{m\omega_3}\right)^{1/2} \Rightarrow A_3 = \left(\frac{m\omega_3}{\pi\hbar}\right)^{1/4}$$

So $A = A_1 A_2 A_3 = \boxed{\left(\dfrac{4m^5\omega_1^3\omega_2\omega_3}{\pi^3\hbar^5}\right)^{1/4}}$

d] $n_x = 1$, $n_y = n_z = 0$. So $\boxed{E = \frac{3}{2}\hbar\omega_1 + \frac{1}{2}\hbar\omega_2 + \frac{1}{2}\hbar\omega_3}$

e] No. We can't find another set of (n_x, n_y, n_z) that give us this energy.

f] $\langle U \rangle = \int_{-\infty}^{\infty} |\psi|^2 \left(\frac{1}{2}m\omega_1^2 x^2 + \frac{1}{2}m\omega_2^2 y^2 + \frac{1}{2}m\omega_3^2 z^2\right)$

We can use the fact that we normalized separately to write this as

$$\int_{-\infty}^{\infty} |X(x)|^2 \frac{1}{2}m\omega_1^2 x^2 dx + \int_{-\infty}^{\infty} |Y|^2 \frac{1}{2}m\omega_2^2 y^2 dy + \int_{-\infty}^{\infty} |Z|^2 \frac{1}{2}m\omega_3^2 z^2 dz$$

$$\int_{-\infty}^{\infty} |X(x)|^2 \frac{1}{2}m\omega_1^2 x^2 dx = A_1^2 \cdot \frac{1}{2}m\omega_1^2 \int_{-\infty}^{\infty} x^4 e^{-\frac{m\omega_1}{\hbar}x^2} dx$$

$$= A_1^2 \cdot \frac{1}{2}m\omega_1^2 \cdot \frac{3}{4}\frac{\sqrt{\pi}}{\left(\frac{m\omega_1}{\hbar}\right)^{5/2}} = \frac{\cancel{2} m^{3/2}\omega_1^{3/2}}{\cancel{\pi^{1/2}}\hbar^{3/2}} \cdot \frac{1}{\cancel{2}}\cancel{m}\omega_1^2 \cdot \frac{3}{4}\frac{\cancel{\pi^{1/2}}\hbar^{5/2}}{\cancel{m^{5/2}}\omega_1^{5/2}}$$

$$= \frac{3}{4}\hbar\omega_1.$$

$$\int_{-\infty}^{\infty} |\Psi(y)|^2 \frac{1}{2} m\omega_2^2 y^2 dy = A_2^2 \frac{1}{2} m\omega_2^2 \int_{-\infty}^{\infty} y^2 e^{-\frac{m\omega_2}{\hbar} y^2} dy$$

$$= \left(\frac{m\omega_2}{\pi\hbar}\right)^{1/2} \frac{1}{2} m\omega_2^2 \cdot \frac{\pi^{1/2}}{2\left(\frac{m\omega_1}{\hbar}\right)^{3/2}}$$

$$= \frac{1}{4}\hbar\omega_2 .$$

The z integral will be the same, but with $\omega_2 \to \omega_3$.

$$\therefore \boxed{\langle U \rangle = \frac{3}{4}\hbar\omega_1 + \frac{1}{4}\hbar\omega_2 + \frac{1}{4}\hbar\omega_3}$$

7a] $n=1,\ \ell=0,\ m=0$ and $Z=3$.

So $\psi(r,\theta,\phi) = R_{10} Y_0^0 = \underbrace{\left(\frac{3}{a_0}\right)^{3/2} 2e^{-3r/a_0}}_{R} \cdot \underbrace{\frac{1}{2\sqrt{\pi}}}_{Y}$.

Since $U = -\frac{k(3e)e}{r}$,

$\langle U \rangle = \langle -\frac{3ke^2}{r} \rangle$. The angular part integrates to 1,

so just do the radial piece:

$$\langle U \rangle = \int_0^\infty \left(-\frac{3ke^2}{r}\right) |R|^2 r^2 dr = -3ke^2 \cdot \left(\frac{3}{a_0}\right)^3 \cdot 4 \int_0^\infty r e^{-6r/a_0} dr$$

$$= -3ke^2 \frac{3^3}{a_0^3} 4 \cdot \left(\frac{a_0}{6}\right)^2$$

$$= -\left(\frac{3^4 \cdot 4}{3^2 \cdot 2^2}\right) \frac{ke^2}{a_0} = \boxed{-\frac{9ke^2}{a_0}}$$

$$E = -\frac{kZ^2e^2}{2a_0} = -\frac{9ke^2}{2a_0}$$

So since $\langle E \rangle = \langle KE \rangle + \langle U \rangle$, $\boxed{\langle KE \rangle = \frac{9ke^2}{2a_0}}$

b) $\langle p \rangle = 0$ by symmetry, and

$$\langle p^2 \rangle = 2m_e \langle KE \rangle = \frac{9 m_e k e^2}{a_0}. \text{ Since } a_0 = \frac{\hbar^2}{m k e^2}$$

$$\langle p^2 \rangle = \frac{9\hbar^2}{a_0^2} \Rightarrow \boxed{\Delta p = \frac{3\hbar}{a_0}}$$

$$\langle r \rangle = \int_0^\infty r|R|^2 r^2 dr = \frac{3^3}{a_0^3}\cdot 4\int_0^\infty r^3 e^{-6r/a_0} dr = \frac{3^3}{a_0^3}\cdot 4\left(\frac{a_0}{6}\right)^4 \cdot 3!$$

$$= \frac{3^4 \cdot 2^3}{3^4 \cdot 2^4} a_0 = \frac{a_0}{2}$$

$$\langle r^2 \rangle = \int_0^\infty r^2 |R|^2 r^2 dr = \frac{3^3}{a_0^3}\cdot 4\left(\frac{a_0}{6}\right)^5 \cdot 4! = \frac{3^4 \cdot 2^5}{3^5 \cdot 2^5} a_0^2 = \frac{a_0^2}{3}$$

So $\Delta r = a_0\sqrt{\frac{1}{3} - \frac{1}{4}} = \boxed{\frac{a_0}{\sqrt{12}}}$

$\therefore \Delta r \Delta p = \frac{3}{\sqrt{12}}\hbar = \boxed{\frac{\sqrt{3}}{2}\hbar}$ Consistent, since $> \frac{\hbar}{2}$.

8a] $s=\frac{1}{2}$, so $|S|=\sqrt{\frac{1}{2}\cdot\frac{3}{2}}\,\hbar=\frac{\sqrt{3}}{2}\hbar$

Now set $|S|=I\omega=\frac{2}{5}m_e r^2\cdot\omega=\frac{2}{5}m_e r^2\cdot\frac{v}{r}=\frac{2}{5}m_e r v$

So $v=\frac{\sqrt{3}}{2}\hbar\frac{1}{\left(\frac{2}{5}\right)m_e r}=\frac{5\sqrt{3}\,\hbar}{4m_e r}=\boxed{278c=8.35\times10^{10}\,\frac{m}{s}}$

b] It's way bigger!

c] Since $v\sim\frac{1}{r}$, v would go $\boxed{\text{up by a factor of }100!}$

It would be $27{,}800c$ instead.

d] $\frac{5\sqrt{3}\,\hbar}{4m_e r}=c \Rightarrow \frac{1}{r}=\frac{4m_e c}{5\sqrt{3}\,\hbar} \Rightarrow r=\frac{5\sqrt{3}\,\hbar}{4m_e c}=8.3\times10^{-13}\,m$

$=\boxed{8.3\times10^{-4}\,nm}$

(or just multiply 3×10^{-6} nm by 278, since $r\sim\frac{1}{v}$).