Lorentz Transformation Between Ref Frames

\[ x' = \gamma (x - vt) \]
\[ y' = y \]
\[ z' = z \]
\[ t' = \gamma \left( t - \frac{vx}{c^2} \right) \]

Inverse Lorentz Transformation

\[ x = \gamma (x' + vt) \]
\[ y = y' \]
\[ z = z' \]
\[ t = \gamma \left( t' + \frac{vx'}{c^2} \right) \]

As \( v \to 0 \), Galilean Transformation is recovered, as per requirement.

Notice: SPACE and TIME Coordinates mixed up !!!
Lorentz Transform for Pair of Events

\[ \Delta x' = \gamma (\Delta x - v \Delta t) \]
\[ \Delta t' = \gamma \left( \Delta t - \frac{v}{c^2} \Delta x \right) \]
\[ \Delta x = \gamma (\Delta x' + v \Delta t') \]
\[ \Delta t = \gamma \left( \Delta t' + \frac{v}{c^2} \Delta x' \right) \]

Can understand Simultaneity, Length contraction & Time dilation formulae from this

Time dilation: Bulb in S frame turned on at \( t_1 \) & off at \( t_2 \) : What \( \Delta t' \) did S’ measure?

- two events occur at same place in S frame => \( \Delta x = 0 \)
- \( \Delta t' = \gamma \Delta t \) (\( \Delta t \) = proper time)

Length Contraction: Ruler measured in S between \( x_1 \) & \( x_2 \) : What \( \Delta x' \) did S’ measure?

- two ends measured at same time in S’ frame => \( \Delta t' = 0 \)
- \( \Delta x = \gamma (\Delta x' + 0) \) => \( \Delta x' = \Delta x / \gamma \) (\( \Delta x \) = proper length)
Fitting a 5m pole in a 4m barnhouse

Student with pole runs with \( v = (3/5)c \)

Farmboy sees pole contraction factor

\[
\sqrt{1 - \left(\frac{3c}{5c}\right)^2} = 4/5
\]

Says pole just fits in the barn fully!

Student with pole runs with \( v = (3/5)c \)

Student sees barn contraction factor

\[
\sqrt{1 - \left(\frac{3c}{5c}\right)^2} = 4/5
\]

Says barn is only 3.2m long, too short to contain entire 5m pole!

Farmboy says “You can do it”

Student says “Dude, you are nuts”

Is there a contradiction? Is Relativity wrong?

Homework: You figure out who is right, if any and why.
Hint: Think in terms of observing three events.
Fitting a 5m pole in a 4m barnhouse?

\[ V = \left( \frac{3}{5} \right) c \]

Answer: Simultaneity!

Event A: Arrival of right end of pole at left end of barn: \((t = 0, t' = 0)\) is reference

\[ L'_0 = \text{proper length of pole in } S' \]

\[ l_0 = \text{length of barn in } S \text{ frame} < L'_0 \]

In \( S \): length of pole \( L = L'_0 \sqrt{1 - (v/c)^2} \)

The times in two frames are related:

\[ t'_B = \frac{l'_B}{v} = \frac{l_0}{v} \sqrt{1 - (v/c)^2} = t_{BC} \sqrt{1 - (v/c)^2} \]

\[ t'_C = \frac{L'_0}{v} = \frac{l'_C}{v} \frac{1}{\sqrt{1 - (v/c)^2}} = \frac{t_{BC}}{\sqrt{1 - (v/c)^2}} \]

⇒ Time gap in \( S' \) by which events B and C fail to be simultaneous

A: Arrival of right end of pole at left end of barn
B: Arrival of left end of pole at left end of barn
C: Arrival of right end of pole at right end of barn

Let \( S = \text{Barn frame}, S' = \text{student frame} \)

Farmboy sees two events as simultaneous

2D student cannot agree

Fitting of the pole in barn is relative!
Farmboy Vs 2D Student

Pole and barn are in relative motion $u$ such that
lorentz contracted length of pole =
Proper length of barn

In rest frame of pole,
Event B precedes C
Lorentz Velocity Transformation Rule

In \( S' \) frame, \( u_{x'} = \frac{x_2' - x_1'}{t_2' - t_1'} = \frac{dx}{dt} \)

\( \quad \frac{dx}{dt} = \gamma(dx - vdt), \quad dt' = \gamma(dt - \frac{v}{c^2}dx) \)

\( u_{x'} = \frac{dx - vdt}{dt} \), divide by \( dt' \)

\[ u_{x'} = \gamma \left( \frac{u - \frac{v}{c}}{1 - \frac{vu}{c^2}} \right) \]

For \( v << c, \ u_{x'} = u_x - v \)

\((Galilean \ Trans. \ Restored)\)

**S and \( S' \) are measuring ant’s speed \( u \) along \( x, y, z \) axes**
Does Lorentz Transform “work”? 

Two rockets travel in opposite directions 

An observer on earth (S) measures speeds = 0.75c And 0.85c for A & B respectively 

What does A measure as B’s speed?

Place an imaginary S’ frame on Rocket A ⇒ v = 0.75c relative to Earth Observer S 

\[
\begin{align*}
    u'_x &= \frac{u_x - v}{\sqrt{1 - \frac{u_x v}{c^2}}} \\
    &= \frac{-0.850c - 0.750c}{\sqrt{1 - \frac{(-0.850c)(0.750c)}{c^2}}} \\
    &= \frac{-1.60c}{\sqrt{1 - \frac{1.5375c^2}{c^2}}} \\
    &= -0.977c
\end{align*}
\]

Consistent with Special Theory of Relativity
Velocity Transformation Perpendicular to S-S’ motion

\[ dy' = dy, \quad dt' = \gamma (dt - \frac{v}{c^2} \, dx) \]

\[ u'_y = \frac{dy'}{dt'} = \frac{dy}{\gamma (dt - \frac{v}{c^2} \, dx)} \]

divide by dt on RHS

\[ u'_y = \frac{u_y}{\gamma (1 - \frac{v}{c^2} u_x)} \]

There is a change in velocity in the direction \( \perp \) to S-S' motion!

Similarly

Z component of Ant's velocity transforms as

\[ u'_z = \frac{u_z}{\gamma (1 - \frac{v}{c^2} u_x)} \]
Inverse Lorentz Velocity Transformation

Inverse Velocity Transform:

\[ u_x = \frac{u_x' + \nu}{1 + \frac{\nu u_x'}{c^2}} \]

\[ u_y = \frac{u_y}{\gamma(1 + \frac{\nu u_x'}{c^2})} \]

\[ u_z = \frac{u_z}{\gamma(1 + \frac{\nu u_x'}{c^2})} \]

As usual, replace \( V \Leftrightarrow -V \)
Example of Inverse velocity Transform

Biker moves with speed $= 0.8c$ past stationary observer

 Throws a ball forward with speed $= 0.7c$

What does stationary observer see as velocity of ball?

Place S’ frame on biker
Biker sees ball speed $u_{X'} = 0.7c$

Speed of ball relative to stationary observer
$u_x$ ?

$u_x = \frac{u_x' + v}{1 + \frac{u_x'v}{c^2}} = \frac{0.70c + 0.80c}{(0.70c)(0.80c)} = 0.96c$
Hollywood Yarns!
Terminator: Can you be seen to be born before your mother?

A frame of Ref where sequence of events is REVERSED ?!!

\[ \Delta t' = t_2' - t_1' = \gamma \left[ \Delta t - \left( \frac{v \Delta x}{c^2} \right) \right] \]

Reversing sequence of events \( \Rightarrow \Delta t' < 0 \)
I Can’t ‘be seen to arrive in SF before I take off from SD

\[ \Delta t' = t'_2 - t'_1 = \gamma \left[ \Delta t - \left( \frac{v \Delta x}{c^2} \right) \right] \]

For what value of \( v \) can \( \Delta t' < 0 \)

\[ \Delta t' < 0 \Rightarrow \Delta t < \frac{v \Delta x}{c^2} \Rightarrow 1 < \frac{v}{c^2 \Delta t} = \frac{v u}{c^2} \]

\[ \Rightarrow \frac{v}{c} > \frac{c}{u} \Rightarrow v > c : \text{Not allowed} \]
Relativistic Momentum and Revised Newton’s Laws

Need to generalize the laws of Mechanics & Newton to confirm to Lorentz Transform and the Special theory of relativity: Example:

\[ p = m \bar{u} \]

\[ \begin{align*}
    v_1 &= 0, \\
    v_2 &= \frac{v - v_1}{\frac{v}{c}} - \frac{2v}{1 - \frac{v^2}{c^2}} \\
    v_1' &= \frac{v}{c^2} - \frac{v}{c} \\
    v_2' &= \frac{v}{c^2} + \frac{v}{c} \\
    p_{\text{before}} &= m v_1 + m v_2 \\
    p_{\text{after}} &= m v_1' + m v_2' \\
    p_{\text{before}} &= p_{\text{after}}
\end{align*} \]

Watching an Inelastic Collision between two putty balls.
Definition (without proof) of Relativistic Momentum

\[ \vec{p} = \frac{m \vec{u}}{\sqrt{1 - (u/c)^2}} = \gamma m \vec{u} \]

With the new definition relativistic momentum is conserved in all frames of references: Do the exercise

New Concepts

Rest mass = mass of object measured
In a frame of ref. where object is at rest

\[ \gamma = \frac{1}{\sqrt{1 - (u/c)^2}} \]

\( u \) is velocity of the object
NOT of a reference frame!
Nature of Relativistic Momentum

\[ \vec{p} = \frac{m\vec{u}}{\sqrt{1 - (u/c)^2}} = \gamma m\vec{u} \]

With the new definition of Relativistic momentum, Momentum is conserved in all frames of references.