# Physics 2D Lecture Slides
# Lecture 23: Feb 23nd 2005

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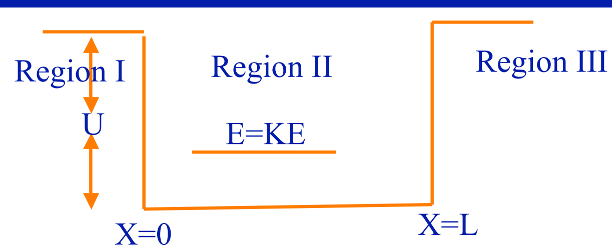
## Finite Potential Barrier

- There are no Infinite Potentials in the real world
  - Imagine the cost of as battery with infinite potential diff
    - Will cost infinite $ sum + not available at Radio Shack
- Imagine a realistic potential : Large U compared to KE but not infinite




Classical Picture : A bound particle (no escape) in $0<x<L$

Quantum Mechanical Picture : Use $\Delta E.\Delta t \leq h/2\pi$

Particle can leak out of the Box of finite potential $P(|x|>L) \neq 0$

## Finite Potential Well

$$\frac{-\hbar^2}{2\text{m}}\frac{d^2\psi(x)}{dx^2} + U\psi(x) = E\ \psi(x)$$

$$\Rightarrow \quad \frac{d^2\psi(x)}{dx^2} = \frac{2m}{\hbar^2}(U-E)\psi(x)$$

$$= \alpha^2\psi(x);\ \alpha = \sqrt{\frac{2\text{m(U-E)}}{\hbar^2}}$$

$\Rightarrow$General Solutions : $\psi(x) = Ae^{+\alpha x} + Be^{-\alpha x}$

Require finiteness of $\psi(x)$

$\Rightarrow \psi(x) = Ae^{+\alpha x}$ .....x<0 (region I)

$\psi(x) = Ae^{-\alpha x}$ .....x>L (region III)

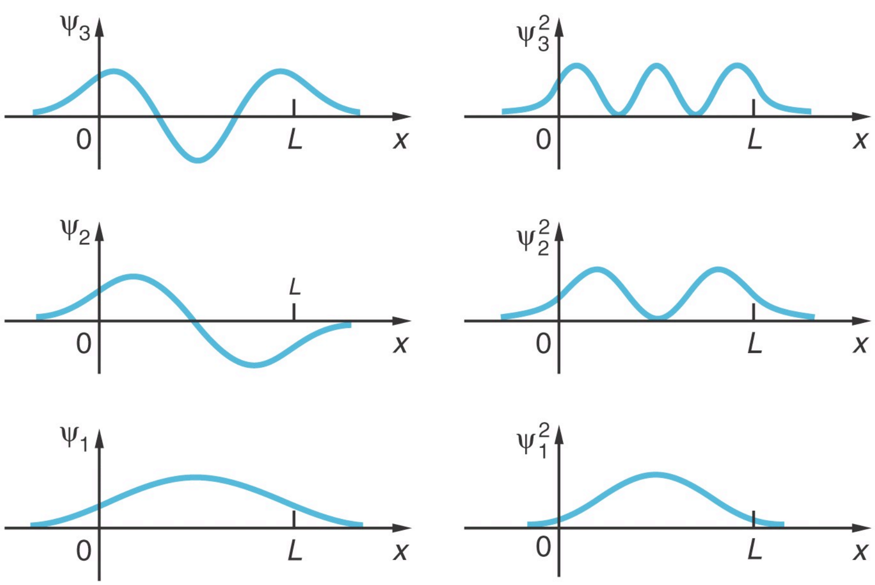
Again, coefficients A & B come from matching conditions at the edge of the walls (x =0, L)

But note that wave fn at $\psi(x)$ at (x =0, L) $\neq 0$ !! (why?)

Further require Continuity of $\psi(x)$ and $\frac{d\psi(x)}{dx}$

These lead to rather different wave functions

## Finite Potential Well: Particle can Burrow Outside Box



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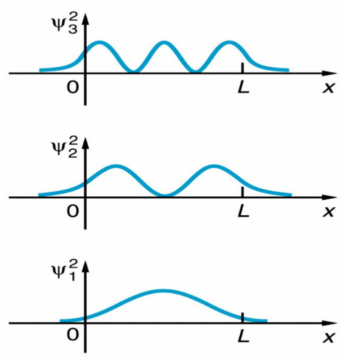


Particle can be outside the box but only for a time $\Delta t \approx h/\Delta E$

$\Delta E$ = Energy particle needs to borrow to Get outside $\Delta E$ = U-E + KE

The Cinderella act (of violating E Conservation cant last very long

Particle must hurry back (cant be caught with its hand inside the cookie-jar)

Penetration Length $\delta = \dfrac{1}{\alpha} = \dfrac{\hbar}{\sqrt{2m(U-E)}}$

If U>>E $\Rightarrow$ Tiny penetration

If $U \rightarrow \infty \Rightarrow \delta \rightarrow 0$

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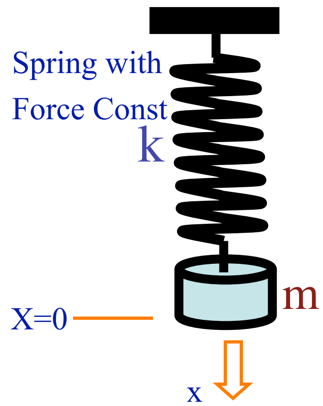
$$E_n = \frac{n^2\pi^2\hbar^2}{2m(L+2\delta)^2}, n = 1,2,3,4...$$

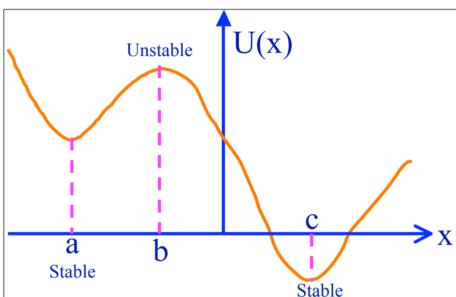
When E=U then solutions blow up

$\Rightarrow$ Limits to number of bound states($E_n < U$)

When E>U, particle is not bound and can get either reflected or transmitted across the potential "barrier"

# Simple Harmonic Oscillator: Quantum and Classical




Particle of mass m within a potential U(x)

$\vec{F}(x) = -\dfrac{dU(x)}{dx}$

$\vec{F}(x=a) = -\left.\dfrac{dU(x)}{dx}\right| = 0,$

$\vec{F}(x=b) = 0$ , $\vec{F}(x=c) = 0$ ...But...

look at the Curvature:

$\dfrac{\partial^2 U}{\partial x^2} > 0$ (stable), $\dfrac{\partial^2 U}{\partial x^2} < 0$ (unstable)

Stable Equilibrium: General Form :

$U(x) = U(a) + \frac{1}{2}k(x-a)^2$

Rescale $\Rightarrow U(x) = \frac{1}{2}k(x-a)^2$

Motion of a Classical Oscillator (ideal)

Ball originally displaced from its equilibrium position, motion confined between x=0 & x=A

$U(x) = \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2x^2; \omega = \sqrt{\frac{k}{m}} = Ang.\ Freq$

$E = \frac{1}{2}kA^2 \Rightarrow$ Changing A changes E

E can take any value & if $A \rightarrow 0$, $E \rightarrow 0$

Max. KE at x = 0, KE= 0 at $x = \pm A$

## Quantum Picture: Harmonic Oscillator

Find the Ground state Wave Function $\psi(x)$

Find the Ground state Energy E when $U(x)=\frac{1}{2}m\omega^2x^2$

Time Dependent Schrodinger Eqn: $\boxed{\frac{-\hbar^2}{2m}\frac{\partial^2\psi(x)}{\partial^2 x}+\frac{1}{2}m\omega^2x^2\psi(x)=E\ \psi(x)}$

$\Rightarrow \boxed{\frac{d^2\psi(x)}{dx^2}=\frac{2m}{\hbar^2}(E-\frac{1}{2}m\omega^2x^2)\psi(x)=0}$ What $\psi(x)$ solves this?

Two guesses about the simplest Wavefunction:

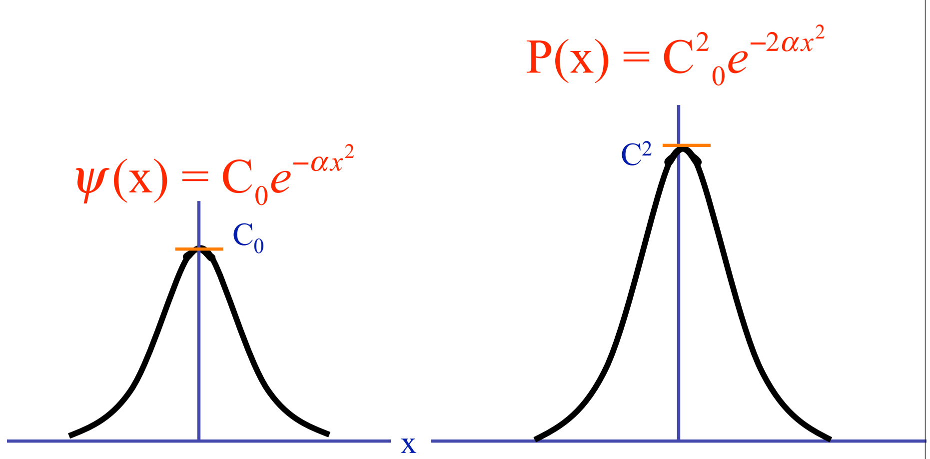
1. $\psi(x)$ should be symmetric about x 2. $\psi(x) \rightarrow 0$ as $x \rightarrow \infty$

$+\ \psi(x)$ should be continuous & $\frac{d\psi(x)}{dx}$ = continuous

My guess: $\psi(x) = C_0 e^{-\alpha x^2}$; Need to find $C_0$ & $\alpha$ :

What does this wavefunction & PDF look like?

## Quantum Picture: Harmonic Oscillator




How to Get $C_0$ & $\alpha$ ?? …Try plugging in the wave-function into the time-independent Schr. Eqn.

## Time Independent Sch. Eqn & The Harmonic Oscillator

Master Equation is : $\frac{\partial^2\psi(x)}{\partial x^2} = \frac{2m}{\hbar^2}[\frac{1}{2}m\omega^2x^2 - E]\psi(x)$

Since $\psi(x) = C_0 e^{-\alpha x^2}$, $\frac{d\psi(x)}{dx} = C_0(-2\alpha x)e^{-\alpha x^2}$,

$$\frac{d^2\psi(x)}{dx^2} = C_0\frac{d(-2\alpha x)}{dx}e^{-\alpha x^2} + C_0(-2\alpha x)^2 e^{-\alpha x^2} = C_0[4\alpha^2x^2 - 2\alpha]e^{-\alpha x^2}$$

$$\Rightarrow C_0[\boxed{4\alpha^2x^2} - \boxed{2\alpha}]e^{-\alpha x^2} = \frac{2m}{\hbar^2}[\boxed{\frac{1}{2}m\omega^2x^2} - E]C_0e^{-\alpha x^2}$$

Match the coeff of $x^2$ and the Constant terms on LHS & RHS

$$\Rightarrow 4\alpha^2 = \frac{2m}{\hbar^2}\frac{1}{2}m\omega^2 \text{ or } \alpha = \frac{m\omega}{2\hbar}$$

& the other match gives $2\alpha = \frac{2m}{\hbar^2}E$, substituing $\alpha \Rightarrow$

$$\boxed{E = \frac{1}{2}\hbar\omega = hf \quad !!!!......(\text{Planck's Oscillators})}$$

What about $C_0$? We learn about that from the Normalization cond.

## SHO: Normalization Condition

$$\int_{-\infty}^{+\infty} |\psi_0(x)|^2\, dx = 1 = \int_{-\infty}^{+\infty} C_0^2 e^{\frac{-m\omega x^2}{\hbar}}\, dx$$

Since $\int_{-\infty}^{+\infty} e^{-ax^2}dx = \sqrt{\frac{\pi}{a}}$ (dont memorize this)

Identifying $a = \frac{m\omega}{\hbar}$ and using the identity above

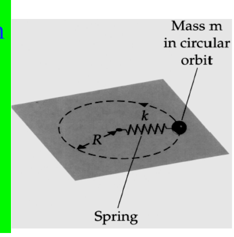
$$\Rightarrow \quad C_0 = \left[\frac{m\omega}{\pi\hbar}\right]^{\frac{1}{4}}$$

Hence the Complete NORMALIZED wave function is :

$$\boxed{\psi_0(x) = \left[\frac{m\omega}{\pi\hbar}\right]^{\frac{1}{4}} e^{\frac{-m\omega x^2}{2\hbar}}}$$ Ground State Wavefunction

has energy E = hf

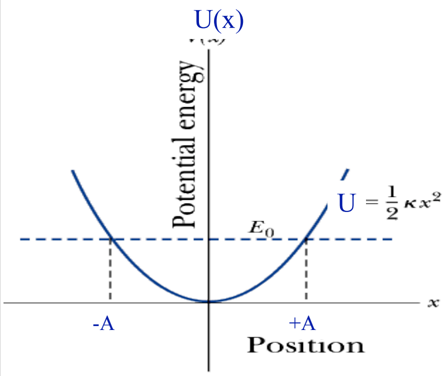
Planck's Oscillators were electrons tied by the "spring" of the mutually attractive Coulomb Force

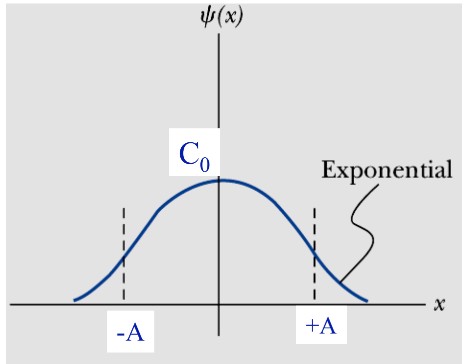


## Quantum Oscillator In Pictures

$E = KE + U(x) > 0$ for n=0

Quantum Mechanical Prob for particle
To live outside classical turning points
Is finite !




Classically particle most likely to be at the turning point (velocity=0)
Quantum Mechanically , particle most likely to be at x=$x_0$ for n=0

## Classical & Quantum Pictures of SHO compared

- Limits of classical vibration : Turning Points (do on Board)
- Quantum Probability for particle outside classical turning points P(|x|>A) =16% !!
  - Do it on the board (see Example problems in book)

## Excited States of The Quantum Oscillator

$\psi_n(x) = C_n H_n(x)\, e^{-\frac{m\omega x^2}{2\hbar}}$ ;

$H_n(x) =$ Hermite Polynomials

with

$H_0(x)=1$

$H_1(x)=2x$

$H_2(x)=4x^2-2$

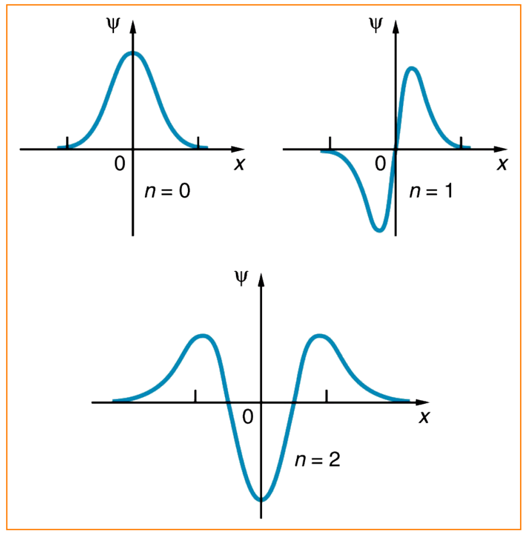
$H_3(x)=8x^3-12x$

$H_n(x)=(-1)^n e^{x^2} \dfrac{d^n e^{-x^2}}{dx^n}$

*and*

$$E_n = (n+\frac{1}{2})\hbar\omega = (n+\frac{1}{2})hf$$

Again n=0,1,2,3...∞ Quantum #

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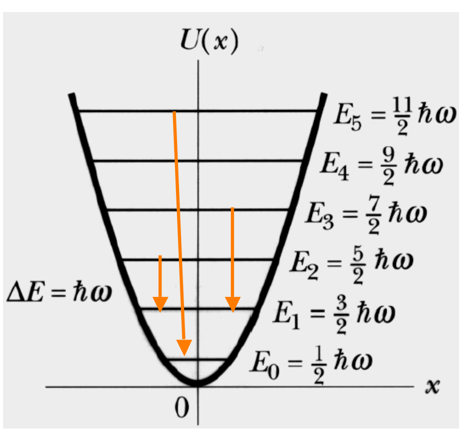


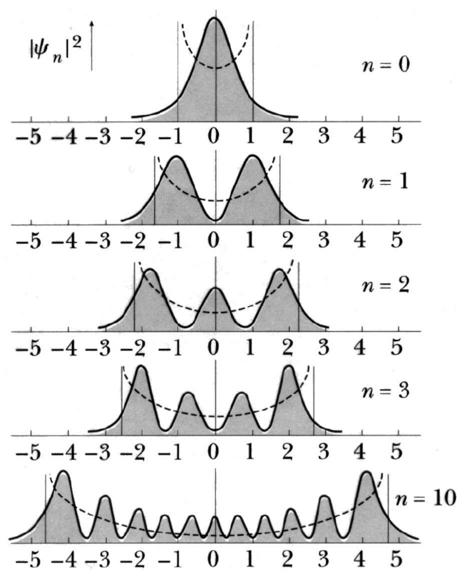


Ground State Energy >0 always