[redacted]

Problem 1 : Particle in a 2D Box With Rigid Walls [10 pts] :

A particle of mass m moves in a two-dimensional box of length L_1 along the X axis and L_2 along the Y axis, with $L_1 < L_2$. One corner of the box lies at the origin, (X=0, Y=0). (a) draw the potential form in 2D. Using the techniques learnt this week (b) write expressions for the wavefunctions and energies as a function of the quantum numbers n_1 and n_2. (c) Write the energies of the ground state and the first excited state (d) Is either of these energy states degenerate? Explain Why (or why not) ?

Problem 2: "Lazy 'R Us" [10 pts]:

Consider a quantum Harmonic oscillator, of mass m under potential $U(x) = \frac{1}{2}m\omega^2 x^2$, in its ground state. For such a system (a) Calculate $\langle x \rangle$, $\langle x^2 \rangle$ and the uncertainty Δx in its location x. (b) Now estimate $\langle p \rangle$ based on symmetry of the potential (c) Write the expression for the total non-relativistic energy for this system and use it to relate $\langle p^2 \rangle$ to $\langle x^2 \rangle$. (d) Finally, calculate the uncertainty Δp and the value of the product $\Delta x \,.\, \Delta p$. By what factor does your calculation agree with Heisenberg's Uncertainty relationship of QM?

Quiz 8

(1) Schrödinger's Eqn in 2-d

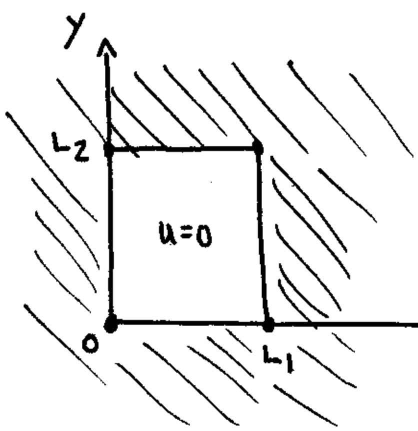
$$-\frac{\hbar^2}{2m}\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}\right)\psi(x,y) + U\psi(x,y) = E\psi(x,y)$$

assume it is seperable

let $\psi(x,y) = \psi_1(x)\psi_2(y)$

then

$$-\frac{\hbar^2}{2m}\left(\psi_2(y)\frac{d^2}{dx^2}\psi_1(x) + \psi_1(x)\frac{d^2}{dy^2}\psi_2(y)\right) = (E-U)\psi_1(x)\psi_2(y)$$



$U=\infty$ in shaded region

inside the well $U=0$, so Schrödinger's Eqn reduces to:

$$-\frac{\hbar^2}{2m}\left(\psi_2\frac{d^2}{dx^2}\psi_1 + \psi_1\frac{d^2}{dy^2}\psi_2\right) = E\psi_1\psi_2$$

divide both sides by $-\frac{\hbar^2}{2m}\psi_1\psi_2$ to obtain

$$\underbrace{\frac{1}{\psi_1}\frac{d^2}{dx^2}\psi_1}_{\text{function only of } x} + \underbrace{\frac{1}{\psi_2}\frac{d^2}{dy^2}\psi_2}_{\text{function only of } y} = \underbrace{-\frac{2mE}{\hbar^2}}_{\text{constant}} = -k^2 \quad , \quad k=\sqrt{\frac{2mE}{\hbar^2}}$$

we thus have 2 independent ordinary differential equations

$$\frac{1}{\psi_1}\frac{d^2}{dx^2}\psi_1 = -k_1^2 \qquad k_1 = \sqrt{\frac{2mE_1}{\hbar^2}}$$

$$\frac{1}{\psi_2}\frac{d^2}{dy^2}\psi_2 = -k_2^2 \qquad k_2 = \sqrt{\frac{2mE_2}{\hbar^2}}$$

$\Rightarrow \quad \psi_1(x) = A\sin(k_1x) + B\cos(k_1x) \qquad \psi_2(y) = C\sin(k_2y) + D\cos(k_2y)$

we can set the phase $\phi = 0$ and drop the cosine since our 2-d well has boundary conditions

$$\psi_1(0) = \psi_1(L_1) = 0$$

$$\psi_2(0) = \psi_2(L_2) = 0$$

evaluating $\psi_1(L_1) = A\sin(K_1 L_1) = 0$

$$\Rightarrow \quad K_1 L_1 = n_1 \pi \qquad (n_1 = 1, 2, \ldots)$$

$$K_1 = \frac{n_1 \pi}{L}$$

$$\psi_2(L_2) = C\sin(K_2 L_2) = 0$$

$$K_2 L_2 = n_2 \pi \qquad (n_2 = 1, 2, \ldots)$$

$$K_2 = \frac{n_2 \pi}{L}$$

we can use the 1-d box normalization to find

$$A = \sqrt{\frac{2}{L_1}}, \quad C = \sqrt{\frac{2}{L_2}}$$

$$\Rightarrow \quad \boxed{\psi(x,y) = \sqrt{\frac{2}{L_1}}\sqrt{\frac{2}{L_2}} \sin\left(\frac{n_1 \pi x}{L_1}\right) \sin\left(\frac{n_2 \pi x}{L_2}\right)}$$

and

$$\boxed{E_{total} = E_1 + E_2 = \frac{\pi^2 \hbar^2}{2m}\left(\frac{n_1^2}{L_1^2} + \frac{n_2^2}{L_2^2}\right)}$$

(c) ground state $n_1 = 1, n_2 = 1$

$$E_{ground} = \frac{\pi^2 \hbar^2}{2m}\left(\frac{1}{L_1^2} + \frac{1}{L_2^2}\right)$$

first excited state: since $L_1 < L_2$ we have $n_1 = 1$, $n_2 = 2$

$$E_{1st} = \frac{\pi^2 \hbar^2}{2m}\left(\frac{1}{L_1^2} + \frac{4}{L_2^2}\right)$$

(d) there is no degeneracy since $L_1 \neq L_2$

(2) $$\psi(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left[-\frac{m\omega}{2\hbar} x^2\right]$$

(a) $$\langle x \rangle = \int_{-\infty}^{+\infty} dx \; x \, \psi^* \psi = \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \int_{-\infty}^{+\infty} dx \; x \exp\left[-\frac{m\omega}{\hbar} x^2\right] = 0$$

symmetric limits   odd function   even function

$$\langle x^2 \rangle = \int_{-\infty}^{+\infty} dx \; x^2 \psi^* \psi = \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \int_{-\infty}^{+\infty} dx \; x^2 \exp\left[-\frac{m\omega}{\hbar} x^2\right]$$

$$= 2 \cdot \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \int_0^{\infty} dx \; x^2 \exp\left[-\frac{m\omega}{\hbar} x^2\right]$$

use

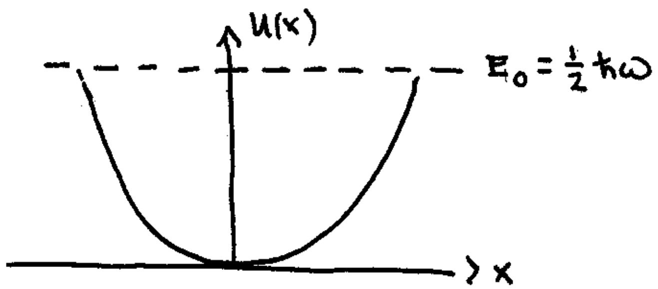
$$\int_0^{\infty} x^2 e^{-ax^2} dx = \frac{1}{4a}\sqrt{\frac{\pi}{a}}$$

w/ $a = \left(\frac{m\omega}{\hbar}\right)$

$$\Rightarrow \langle x^2 \rangle = 2\left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \frac{\hbar}{4m\omega}\sqrt{\frac{\pi\hbar}{m\omega}} = \frac{\hbar}{2m\omega}$$

$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{\langle x^2 \rangle} = \sqrt{\frac{\hbar}{2m\omega}}$$

(b)



when the particle reaches the turning point in its oscillation cycle, its momentum is zero. This means that the sign of its momentum changes. Since the particle moves back and forth between its turning points, and the potential is symmetric, the momentum is to the left the same amount of time it is to the right, so $\langle p_x \rangle = 0$.

(c) $$E_0 = \frac{1}{2}\hbar\omega = \frac{1}{2}m\omega^2\langle x^2\rangle + \frac{\langle p_x^2\rangle}{2m} = U + K$$

$$\Rightarrow \quad \frac{\langle p_x^2\rangle}{2m} = \frac{1}{2}\hbar\omega - \frac{1}{2}m\omega^2\langle x^2\rangle$$

$$\langle p_x^2\rangle = m\hbar\omega - m^2\omega^2\langle x^2\rangle$$

$$= m\hbar\omega - \frac{m^2\omega^2\hbar}{2m\omega} = m\hbar\omega - \frac{1}{2}m\hbar\omega$$

$$= \frac{1}{2}m\hbar\omega$$

$$\Rightarrow \quad \Delta p_x = \sqrt{\langle p_x^2\rangle - \langle p_x\rangle^2} = \sqrt{\frac{1}{2}m\hbar\omega}$$

and $$\Delta p_x \Delta x = \sqrt{\frac{1}{2}m\hbar\omega}\sqrt{\frac{\hbar}{2m\omega}} = \frac{\hbar}{2}$$

this is the minimum uncertainty since $\Delta p_x \Delta x \geq \hbar/2$