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Modern Physics (2-D)
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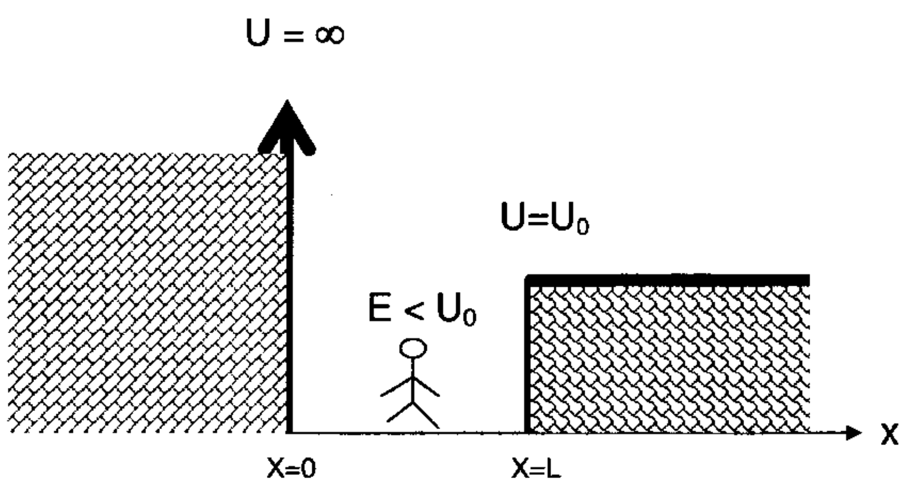
Quiz # 7 (Nov 21)

Problem 1 : Drawing Lesson ! [12 pts] : Consider a particle moving in a one-dimensional box with rigid walls ($U=\infty$) at $x= -L/4$ and $x=L/4$. (a) Sketch the potential in all regions of x , (b) write the wave functions and probability densities for the states n=1, 2, 3 (c) sketch the wave function and the probability density in all regions of x, (d) Calculate the ground state (n=1) energy of the particle.

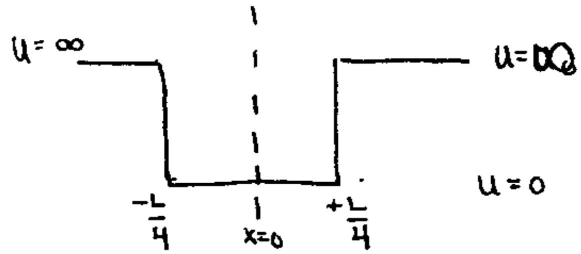
Problem 2: Not Just Another Brick in the Wall ! [8 pts]: Suppose that a particle of mass m is trapped in a potential well that has rigid wall at $x = 0$ ($U = \infty$ for $x < 0$) and a finite wall of height $U=U_0$ at $x = L$. See figure below and assume particle energy $E < U_0$. (a) Cleanly sketch the wave function for the lowest three states and be sure to label points x=0 & x=L on your sketches, (b) What is the mathematical form of the wave function in the ground state in the three regions $x < 0$, $0 < x < L$ and $X > L$?



(i)

QUIZ 7

(a)



(b)

wavefunction must vanish for $x > L/4$ and $x < -L/4$, and the wavefunction must also be continuous everywhere, so

$$\psi(+L/4) = \psi(-L/4) = 0$$

solve schrödinger's eqn:

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi + U\psi = E\psi$$

inside the well, $U=0$

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi = E\psi$$

$$\frac{d^2}{dx^2}\psi = -\frac{2mE}{\hbar^2}\psi = -k^2\psi, \quad k = \sqrt{\frac{2mE}{\hbar^2}}$$

and the solutions to the differential eqn are

$$\psi(x) = A\sin(kx+\phi) + B\cos(kx+\phi)$$

let's impose the conditions $\psi(-L/4) = \psi(+L/4) = 0$

$$\psi(-L/4) = A\sin\left(-\frac{kL}{4}+\phi\right) + B\cos\left(-\frac{kL}{4}+\phi\right) = 0$$

let's choose $\phi = +\frac{kL}{4}$, then

$$\psi(-L/4) = A\sin\left(-\frac{kL}{4}+\frac{kL}{4}\right) + B\cos\left(-\frac{kL}{4}+\frac{kL}{4}\right)$$

$$= 0$$

since $\sin(0) = 0$, $\cos(0) = 1$

we have

$$B = 0$$

so $\psi(x) = A\sin(kx + kL/4)$

now $\quad \psi(+L/4) = 0 = A \sin\left(\frac{KL}{4} + \frac{KL}{4}\right) = A \sin\left(\frac{KL}{2}\right)$

for sin to vanish, we must have

$$\frac{KL}{2} = n\pi \quad \Rightarrow \quad K = \frac{2n\pi}{L}$$

$$\Rightarrow \quad \psi(x) = A \sin\left(kx + kL/4\right)$$

$$= A \sin\left[k\left(x + L/4\right)\right]$$

$$\boxed{\psi_n(x) = A \sin\left[\frac{2n\pi}{L}\left(x + L/4\right)\right]}$$

(you could have also arrived at $(x - L/4)$ depending upon the order in which you imposed the boundary conditions).

Now let's find the normalization constant:

$$A^2 \int_{-L/4}^{+L/4} dx \ \sin^2\left[\frac{2n\pi}{L}\left(x + L/4\right)\right] = 1$$

$$\text{let} \quad u = \frac{2n\pi}{L}\left(x + L/4\right)$$

$$du = \frac{2n\pi}{L} dx$$

$$\text{so} \quad dx = \frac{L}{2n\pi} du$$

limits change to $\int_0^{n\pi}$

$$\frac{LA^2}{2n\pi} \int_0^{n\pi} du \ \sin^2 u = \frac{LA^2}{2n\pi}\left[\frac{1}{2}u - \frac{1}{4}\sin 2u\right]\Bigg|_0^{n\pi}$$

$$= \frac{LA^2}{4} = 1$$

$$\Rightarrow \quad A = \sqrt{\frac{4}{L}}$$

$$\boxed{\psi_n(x) = \sqrt{\frac{4}{L}} \ \sin\left[\frac{2n\pi}{L}\left(x + L/4\right)\right]}$$

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$$\Rightarrow \quad \psi_n(x) = \sqrt{\frac{4}{L}} \; \sin\left[\frac{2n\pi x}{L} + \frac{n\pi}{2} \right]$$

$$\psi_n(x) = \sqrt{\frac{4}{L}} \left[\sin\left(\frac{2n\pi x}{L}\right) \cos\left(\frac{n\pi}{2}\right) + \cos\left(\frac{2n\pi x}{L}\right) \sin\left(\frac{n\pi}{2}\right) \right]$$

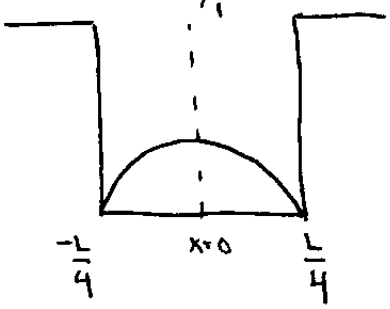
for $n=1$ $\quad \psi_1(x) = \sqrt{\frac{4}{L}} \cos\left(\frac{2\pi x}{L}\right)$

$n=2$ $\quad \psi_2(x) = \sqrt{\frac{4}{L}} \sin\left(\frac{4\pi x}{L}\right)$ $\quad$ (forget about minus sign)

$n=3$ $\quad \psi_3(x) = \sqrt{\frac{4}{L}} \cos\left(\frac{6\pi x}{L}\right)$ $\quad$ (forget about minus sign again)

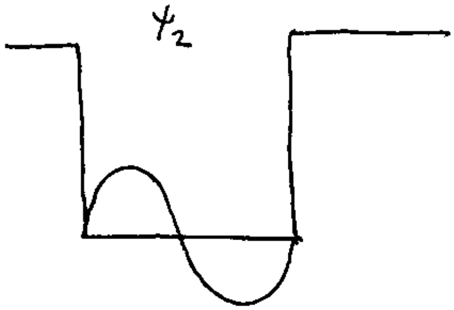
the probability densities are just $|\psi_n|^2$

$n=1$ $\quad \psi_1$

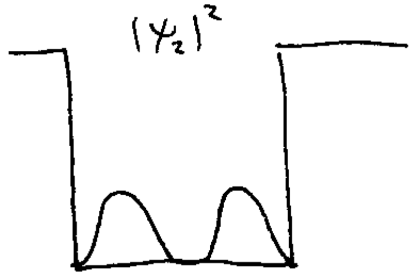


$|\psi_1|^2$

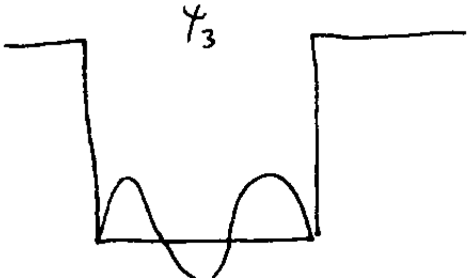
ψ_2



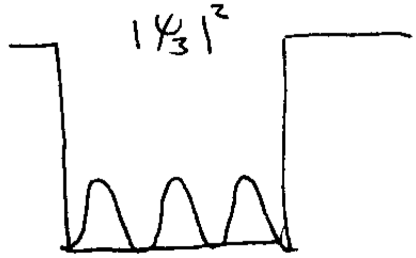
$|\psi_2|^2$



ψ_3



$|\psi_3|^2$

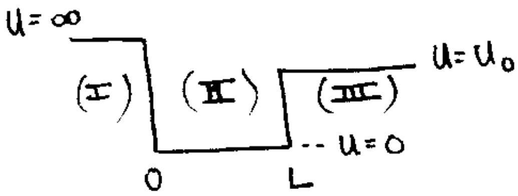


(d) the width of the well is $\frac{L}{2}$ so the $n=1$ energy is just

$$E_1 = \frac{\pi^2 \hbar^2}{2m\left(\frac{L}{2}\right)^2}$$

$$= \frac{2\pi^2 \hbar^2}{m L^2}$$

(2)



$$\boxed{\psi^{(I)}(x) = 0}$$

the wavefunction must vanish in region (I) since the potential is infinite here, so we must satisfy the condition $\psi(0)=0$ for the wavefunction to be continuous.

in region (II), schrödinger's eqn reads
$(0<x<L)$

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi = E\psi \quad \text{since} \quad U=0 \quad \text{here}$$

$$\frac{d^2}{dx^2}\psi = -\frac{2mE}{\hbar^2}\psi = -K^2\psi$$

$$\Rightarrow \psi^{(II)}(x) = A\sin(Kx+\phi) + B\cos(Kx+\phi)$$

impose condition $\psi^{(II)}(0) = 0$

$$\Rightarrow \psi^{(II)}(0) = A\sin\phi + B\cos\phi = 0$$

choose $\phi=0$, and we obtain

$$B=0$$

$$\Rightarrow \boxed{\psi^{(II)}(x) = A\sin Kx}$$

in region (III) $(x>L)$, schrödinger's eqn is

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi + U_0\psi = E\psi \qquad , U=U_0$$

$$\frac{d^2}{dx^2}\psi = \frac{2m(U_0-E)}{\hbar^2}\psi$$

now $E<U_0$ so $\frac{2m(U_0-E)}{\hbar^2} > 0$

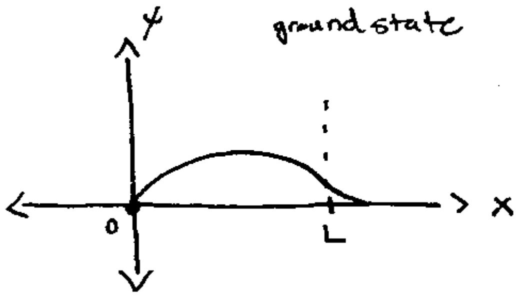
$$\Rightarrow \frac{d^2}{dx^2}\psi = \alpha^2\psi \qquad \alpha = \sqrt{\frac{2m(U_0-E)}{\hbar^2}}$$

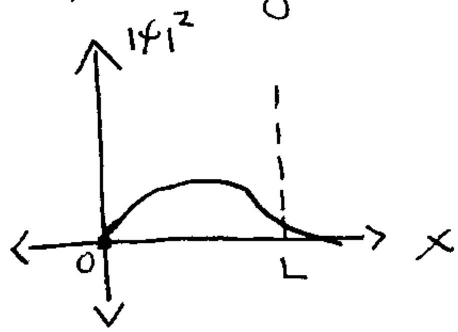
this has solutions $\psi^{(III)}(x) = Ae^{\alpha x} + Be^{-\alpha x}$

but as $x \to \infty$ $e^{\alpha x} \to \infty$, so we set $A = 0$, since the wavefunction must be finite everywhere.

$$\Rightarrow \boxed{\psi^{(III)}(x) = Be^{-\alpha x}}$$

so the wavefunctions and their probability densities look like





you did not have to sketch $|\psi|^2$