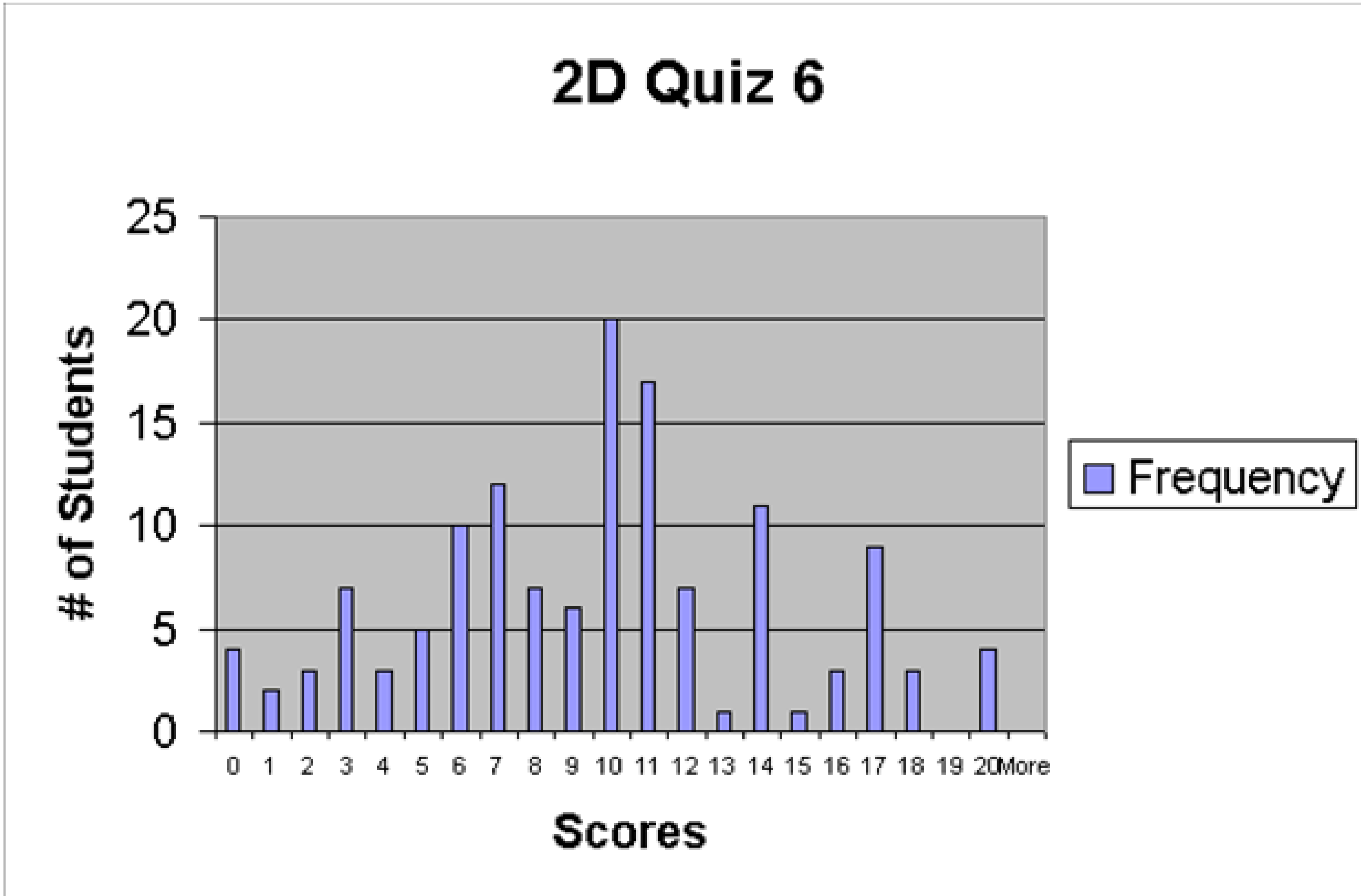
Problem 1 was straight from the HW

Problem 2 was based on solved problem 4.11 in book



# Physics 2D Lecture Slides
# Nov 17

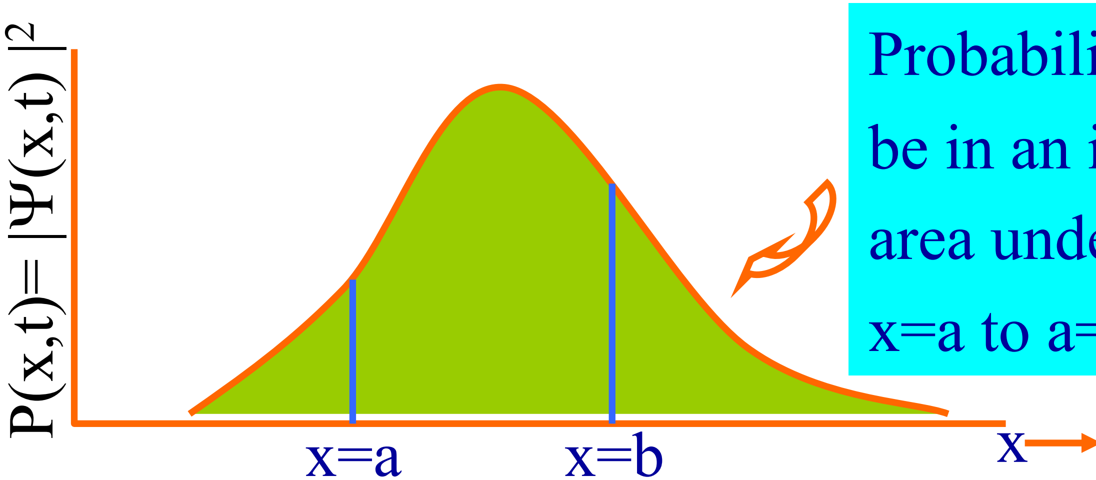
Vivek Sharma
UCSD Physics

- Tuesday Office Hours moved
  - from 2:30-3:30pm Tuesday
  - To → **4:00-6:00pm** ← Tuesday
- Today’s office hour unchanged at 2:00pm, 3314 Mayer Hall

# Quantum Mechanics of Subatomic Particles

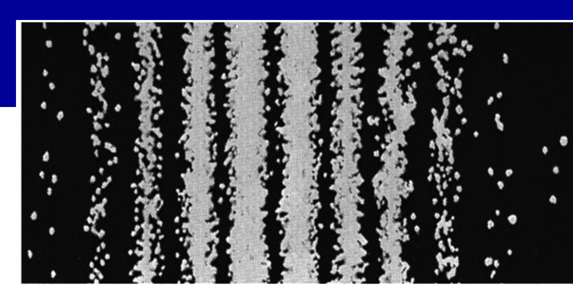
- Act of Observation perturbs the system (No watching!)
- If can't watch then All conversations can only be in terms of a probability P
- Every particle under the influence of a force is described by a Complex wave function $\Psi = \Psi(x,y,z,t)$
- $\Psi$ is the ultimate DNA of particle: contains all info about the particle under the force (e.g. Coulomb force in Hydrogen atom)
- Probability of per unit volume of finding the particle at some point (x,y,z) and at time t is given by
  - $P(x,y,z,t) = \Psi(x,y,z,t) \,.\, \Psi^{*}(x,y,z,t) = |\Psi(x,y,z,t)|^2$
- When there are more than one path to reach a final location then the probability of the event is
  - When there are two paths: $\Psi = \Psi_1 + \Psi_2$
  - $P = |\Psi^{*}\Psi| = |\Psi_1|^2 + |\Psi_2|^2 + 2|\Psi_1||\Psi_2|\cos\phi$

# Wave Function of "Stuff" & Probability Density




Probability of a particle to be in an interval $a \le x \le b$ is area under the curve from x=a to a=b

- Although not possible to specify with certainty the location of particle, its possible to assign probability P(x)dx of finding particle between x and x+dx
- $P(x)\,dx = |\,\Psi(x,t)|^2\,dx$
- e.g intensity distribution in light diffraction pattern is a measure of the probability that a photon will strike a given point within the pattern



# Ψ: The Wave function Of A Particle

- The particle must be some where

$$\int_{-\infty}^{+\infty} |\psi(x,t)|^2 \, dx = 1$$

- Any Ψ satisfying this condition is NORMALIZED
- Prob of finding particle in finite interval

$$P(a \le x \le b) = \int_a^b \psi^*(x,t) \; \psi(x,t) dx$$

- Fundamental aim of Quantum Mechanics
  - Given the wavefunction at some instant (say t=0) find Ψ at some subsequent time t
  - Ψ(x,t=0) → Ψ(x,t) …evolution
  - Think of a probabilistic view of particle's "newtonian trajectory"
    - We are replacing Newton's 2nd law for subatomic systems

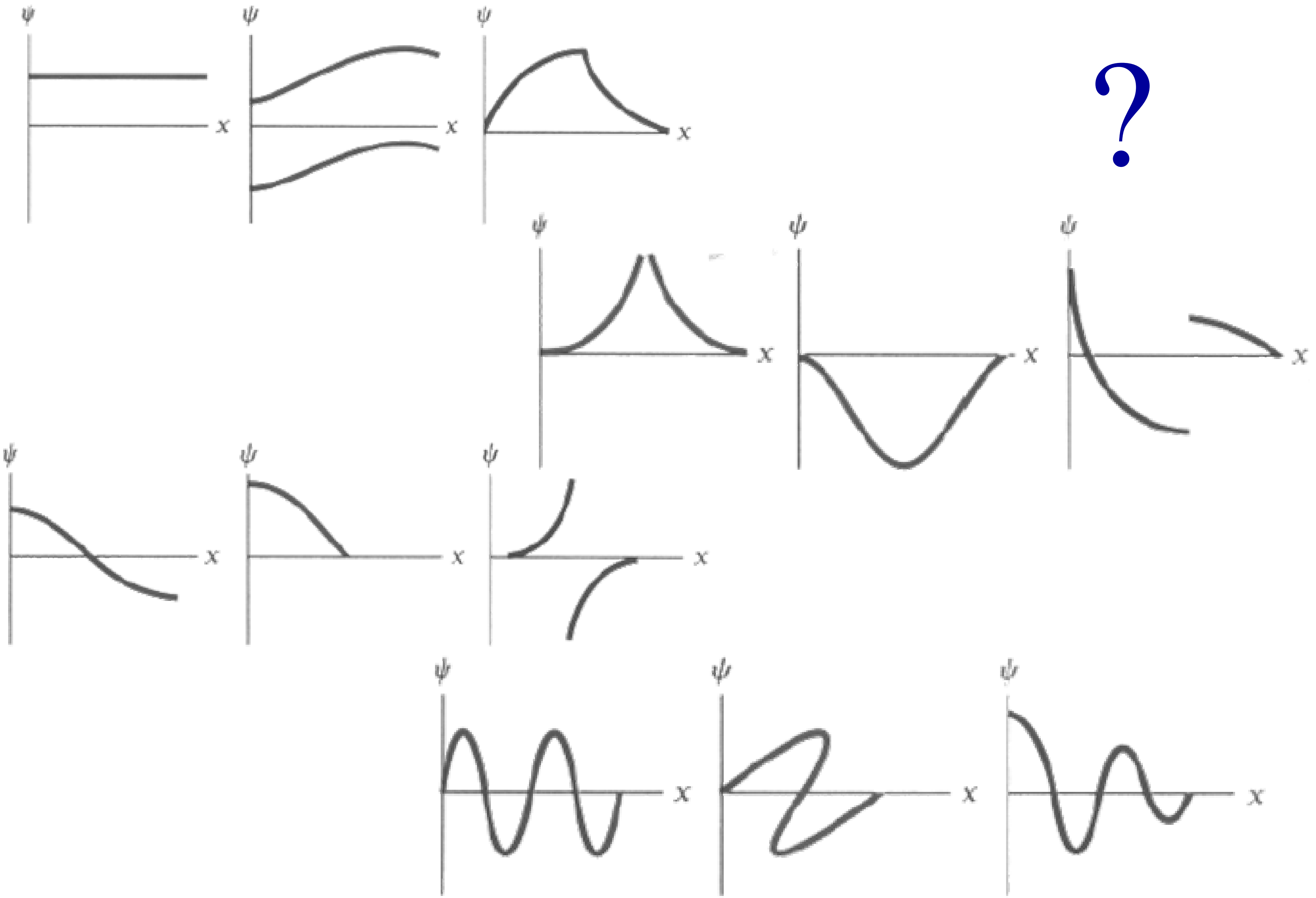
The Wave Function is a mathematical function that describes a physical object → Wave function must have some rigorous Physical properties :

- Ψ must be finite in x,t
- Ψ must be continuous fn of x,t
- Ψ must be single-valued
- Ψ must be smooth fn →

$\frac{d\psi}{dx}$ must be continuous

WHY ?

# Bad (Mathematical) Wave Functions Of a Physical System : You Decide Why



# A Simple Wave Function : Free Particle

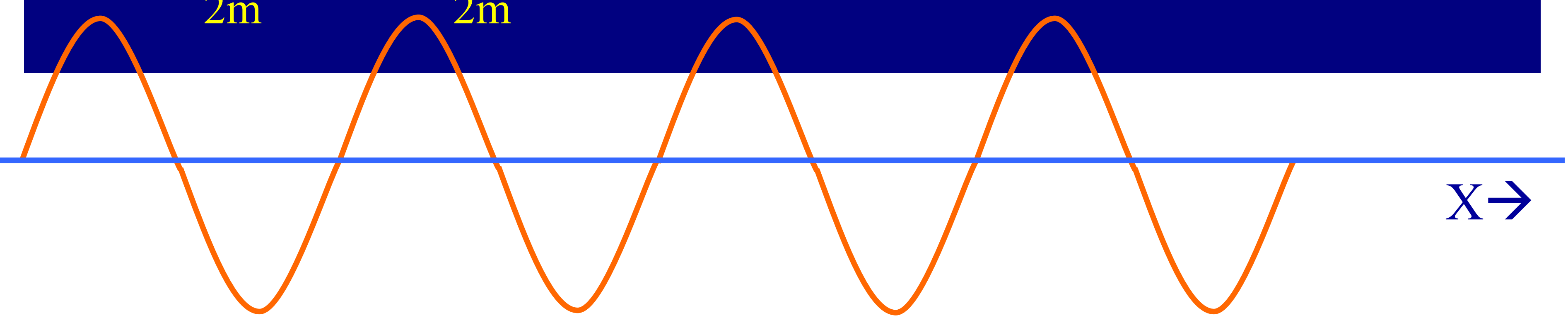
- Imagine a free particle of mass m , momentum p and $K=p^2/2m$
- Under no force , no attractive or repulsive potential to influence it
- Particle is where it wants : can be any where $[-\infty \le x \le +\infty]$
  - Has No relationship, no mortgage , no quiz, no final exam….its essentially a **bum** !
  - how to describe a **quantum mechanical bum** ?
    - $\Psi(x,t) = Ae^{i(kx-\omega t)} = A(\cos(kx-\omega t) + i \sin(kx-\omega t))$

$$k = \frac{p}{\hbar}; \quad \omega = \frac{E}{\hbar}$$

For non-relativistic particles

$$E = \frac{p^2}{2m} \Rightarrow \omega(k) = \frac{\hbar k^2}{2m}$$

Has definite momentum and energy but location unknown !



# Wave Function of Different Kind of Free Particle : Wave Packet

Sum of Plane Waves:

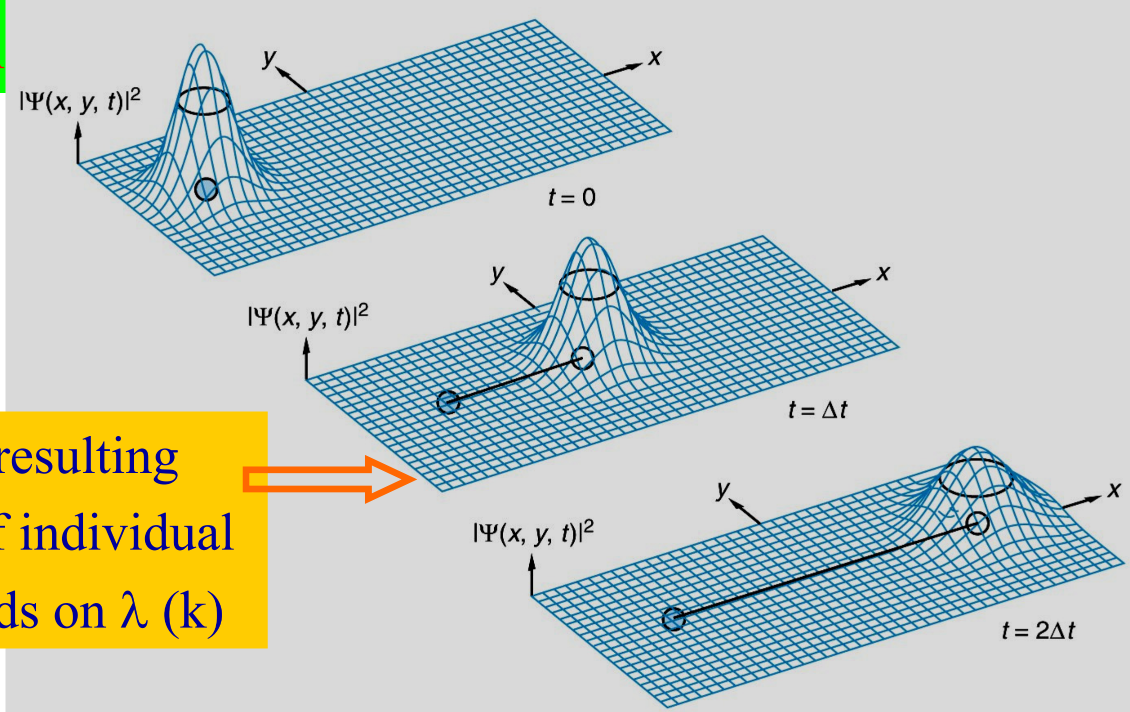
$$\Psi(x,0) = \int_{-\infty}^{+\infty} a(k) e^{ikx} dk$$

$$\Psi(x,t) = \int_{-\infty}^{+\infty} a(k) e^{i(kx-\omega t)} dk$$

Wave Packet initially localized in $\Delta X$, $\Delta t$ undergoes dispersion

Combine many free waves to create a Localized wave packet (group)

The more you know now,
The less you will know later
Why ?




Spreading is due to DISPERSION resulting from the fact that phase velocity of individual waves making up the packet depends on λ (k)

# Normalization Condition: Particle Must be Somewhere

$Example: \ \psi(x,0) = Ce^{-\left|\frac{x}{x_0}\right|}$, C & $x_0$ are constants

This is a symmetric wavefunction with diminishing amplitude

The Amplitude is maximum at x =0 $\Rightarrow$ Probability is max too

Normalization Condition: How to figure out C ?

A real particle must be somewhere: Probability of finding

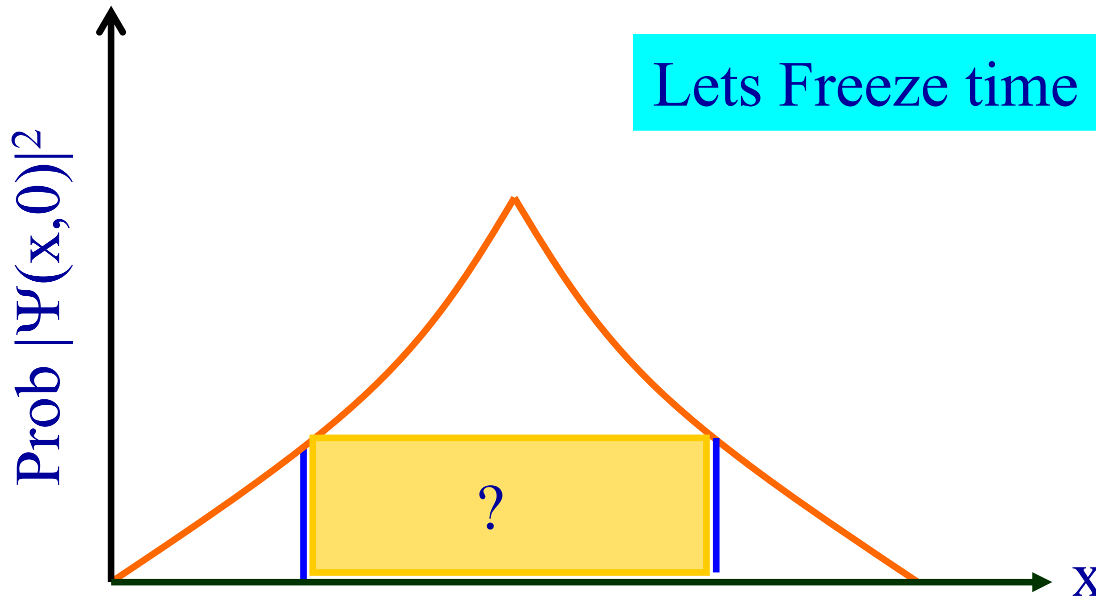
particle is finite $P(-\infty \le x \le +\infty) = \int_{-\infty}^{+\infty} |\psi(x,0)|^2 dx = \int_{-\infty}^{+\infty} C^2 e^{-2\left|\frac{x}{x_0}\right|} dx = 1$

$$\Rightarrow 1 = 2C^2 \int_0^{\infty} e^{-2\left|\frac{x}{x_0}\right|} dx = 2C^2 \left[\frac{x_0}{2}\right] = C^2 x_0$$

$$\Rightarrow \boxed{\psi(x,0) = \frac{1}{\sqrt{x_0}} e^{-\left|\frac{x}{x_0}\right|}}$$

# Where is the particle within a certain location $x \pm \Delta x$

Lets Freeze time (t=0)




$$P(-x_0 \le x \le +x_0) = \int_{-x_0}^{+x_0} |\psi(x,0)|^2 dx = \int_{-x_0}^{+x_0} C^2 e^{-2\left|\frac{x}{x_0}\right|} dx$$

$$= 2C^2\left[\frac{x_0}{2}\right]\left[1-e^{-2}\right] = \left[1-e^{-2}\right] = 0.865 \Rightarrow 87\%$$

# Where Do Wave Functions Come From ?

- Are solutions of the time dependent Schrödinger Differential Equation (inspired by Wave Equation seen in 2C)

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi(x,t)}{\partial x^2}+U(x)\Psi(x,t)=i\hbar\frac{\partial\Psi(x,t)}{\partial t}$$

- Given a potential U(x) → particle under certain force

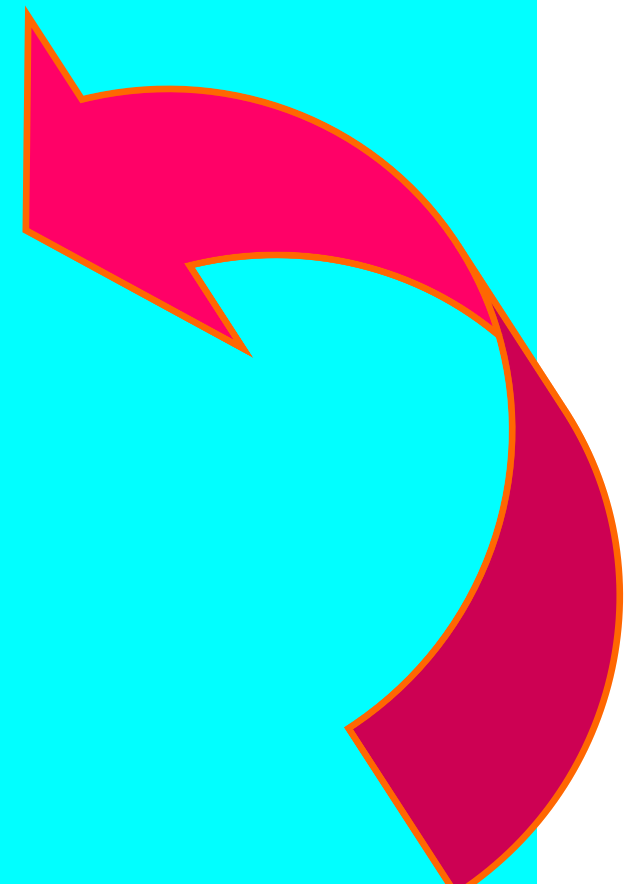
$$-\ F(x) = \ -\frac{\partial U(x)}{\partial x}$$

Schrodinger had an interesting life

# Schrodinger Wave Equation

Wavefunction $\psi$ which is a sol. of the Sch. Equation embodies all modern physics experienced/learnt so far:

$E=hf, \quad p=\frac{h}{\lambda}, \quad \Delta x.\Delta p \sim \hbar, \; \Delta E.\Delta t \sim \hbar,$ quantization etc



Schrodinger Equation is a Dynamical Equation much like Newton's Equation $\vec{F} = m\,\vec{a}$

$$\psi(x,0) \rightarrow \vec{\text{Force}}\,(\text{potential}) \rightarrow \psi(x,t)$$

Evolves the System as a function of space-time

The Schrodinger Eq. propogates the system forward & backward in time:

$$\psi(x,\delta t) = \psi(x,0) \pm \left[\frac{d\psi}{dt}\right]_{t=0} \delta t$$

Where does it come from ?? ..."First Principles"..no real derivation exists

# Time Independent Sch. Equation

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi(x,t)}{\partial x^2}+U(x)\Psi(x,t)=i\hbar\frac{\partial\Psi(x,t)}{\partial t}$$

Sometimes (depending on the character of the Potential U(x,t))

The Wave function is factorizable: can be broken up

$$\Psi(\mathrm{x,t})=\psi(x)\ \phi(t)$$

*Example*: Plane Wave $\Psi(\mathrm{x,t})=e^{i(kx-\omega t)}=e^{i(kx)}e^{-i(\omega t)}$

In such cases, use seperation of variables to get :

$$\frac{-\hbar^2}{2\mathrm{m}}\phi(t).\frac{\partial^2\psi(x)}{\partial^2 x}+U(x)\psi(x)\phi(t)=i\hbar\psi(x)\frac{\partial\phi(t)}{\partial t}$$

Divide Throughout by $\Psi(\mathrm{x,t})=\psi(\mathrm{x})\phi(\mathrm{t})$

$$\Rightarrow \frac{-\hbar^2}{2\mathrm{m}}\frac{1}{\psi(x)}.\frac{\partial^2\psi(x)}{\partial^2 x}+U(x)=i\hbar\frac{1}{\phi(t)}\frac{\partial\phi(t)}{\partial t}$$

LHS is a function of x; RHS is fn of t

x and t are independent variables, hence :

$\Rightarrow$ RHS = LHS = Constant = E

# Factorization Condition For Wave Function Leads to:

$$\frac{-\hbar^2}{2m}\frac{\partial^2\psi(x)}{\partial^2 x}+U(x)\psi(x)=E\;\psi(x)$$

$$i\hbar\frac{\partial\phi(t)}{\partial t}=E\phi(t)$$

What is the Constant E ? How to Interpret it ?

Back to a Free particle :

$\Psi(x,t)= Ae^{ikx}e^{-i\omega t}$, $\psi(x)= Ae^{ikx}$

$U(x,t) = 0$

Plug it into the Time Independent Schrodinger Equation (TISE) $\Rightarrow$

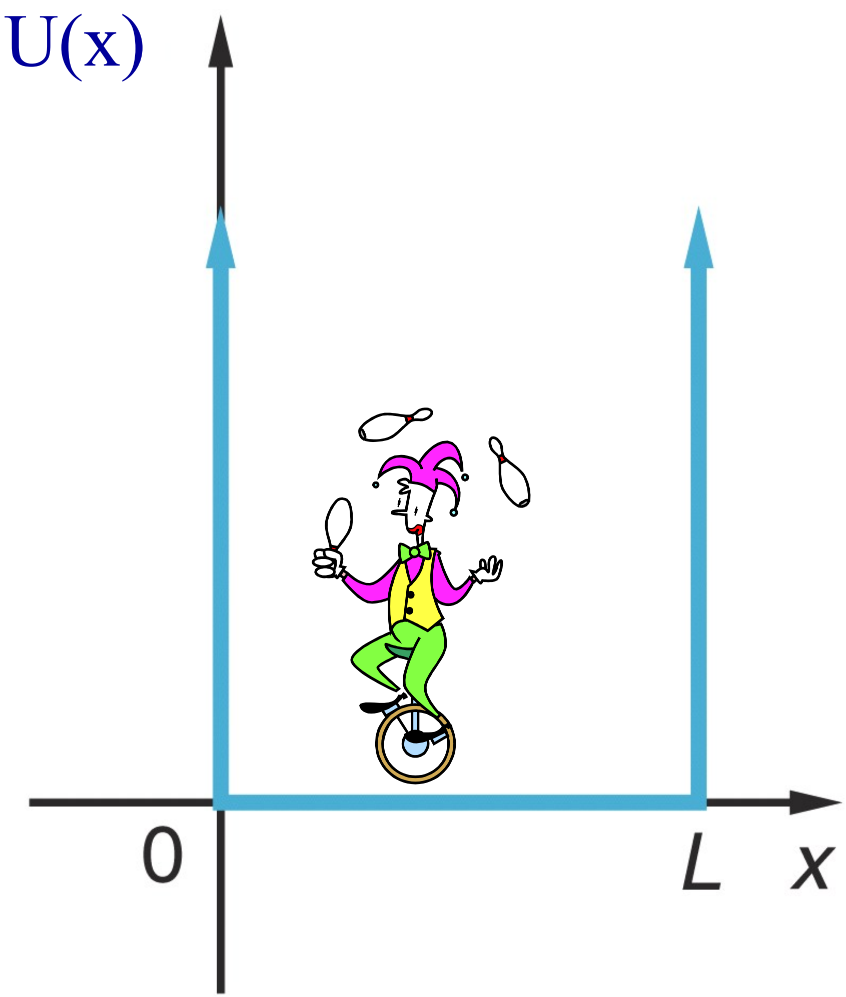
$$\frac{-\hbar^2}{2m}\frac{d^2(Ae^{(ikx)})}{dx^2}+0=E\;Ae^{(ikx)} \;\;\Rightarrow E=\frac{\hbar^2k^2}{2m}=\frac{p^2}{2m}=\text{(NR Energy)}$$

Stationary states of the free particle: $\Psi(x,t)=\psi(x)e^{-i\omega t}$

$$\Rightarrow \left|\Psi(x,t)\right|^2=\left|\psi(x)\right|^2$$

Probability is static in time t, character of wave function depends on $\psi(x)$

# A More Interesting Potential : Particle In a Box




Write the Form of Potential: Infinite Wall

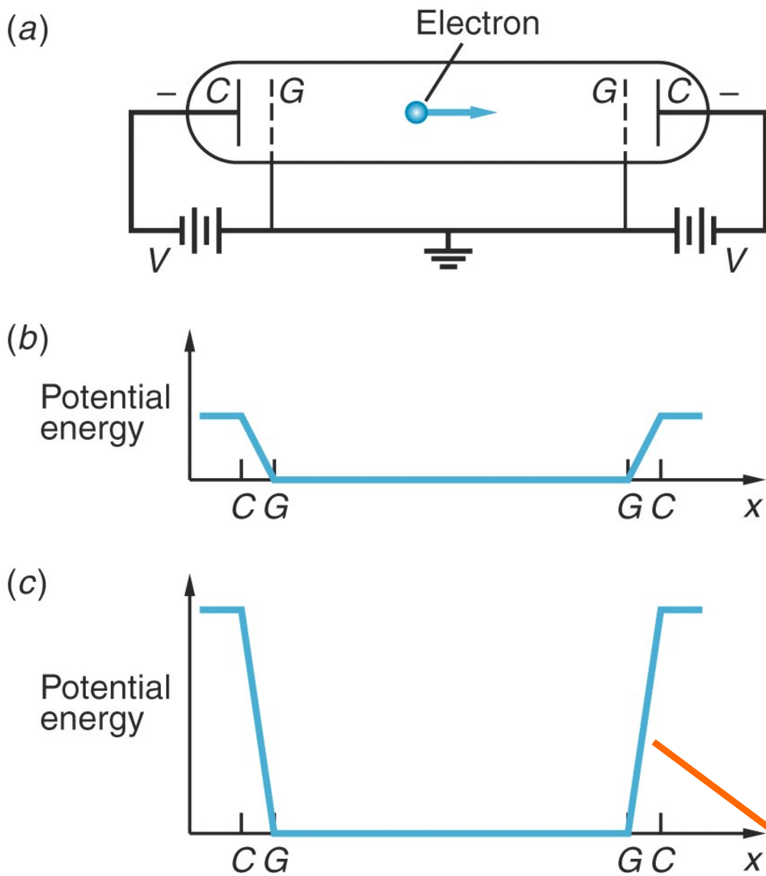
$U(x,t) = \infty; \ x \le 0, \ x \ge L$

$U(x,t) = 0 \ ; \ 0 < X < L$

- Classical Picture:
  - Particle dances back and forth
  - Constant speed, const KE
  - Average <P> = 0
  - No restriction on energy value
    - E=K+U = K+0
  - Particle can not exist outside box
    - Can't get out because needs to borrow infinite energy to overcome potential of wall

What happens when the joker is subatomic in size ??

# Example of a Particle Inside a Box With Infinite Potential



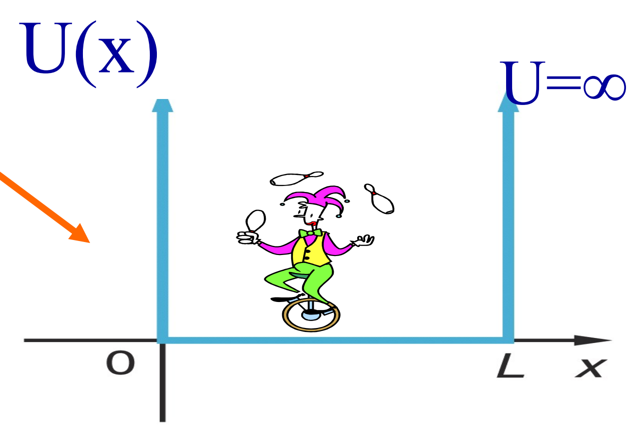

(a) Electron placed between 2 set of electrodes C & grids G experiences no force in the region between grids, which are held at Ground Potential

However in the regions between each C & G is a repelling electric field whose strength depends on the magnitude of V

(b) If V is small, then electron's potential energy vs x has low sloping "walls"

(c) If V is large, the "walls" become very high & steep becoming infinitely high for $V \rightarrow \infty$

(d) The straight infinite walls are an approximation of such a situation



# Ψ(x) for Particle Inside 1D Box with Infinite Potential Walls

Inside the box, no force $\Rightarrow$ U=0 or constant (same thing)

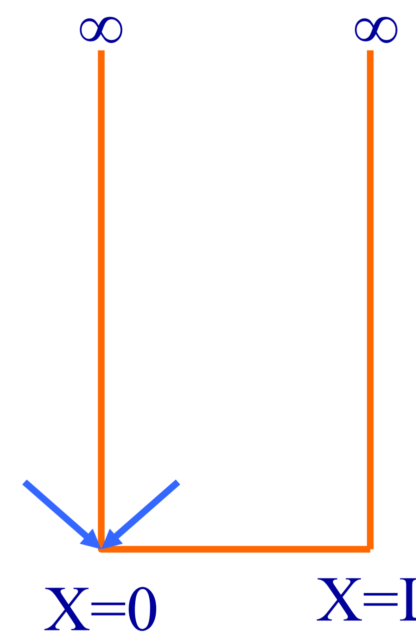
$$\Rightarrow \frac{-\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + 0\ \psi(x) = E\ \psi(x)$$

$$\Rightarrow \frac{d^2\psi(x)}{dx^2} = -k^2\psi(x)\ ;\ k^2 = \frac{2mE}{\hbar^2}$$

*or* $\frac{d^2\psi(x)}{dx^2} + k^2\psi(x) = 0$ $\Leftarrow$ figure out what $\psi$(x) solves this diff eq.

Why can't the particle exist Outside the box ?

→ E Conservation




In General the solution is $\psi(x) = A\ sinkx + B\ coskx$ (A,B are constants)

Need to figure out values of A, B : How to do that ?

Apply BOUNDARY Conditions on the Physical Wavefunction

We said $\psi(x)$ must be continuous everywhere

So match the wavefunction just outside box to the wavefunction value just inside the box

$\Rightarrow$ At x = 0 $\Rightarrow \psi(x=0) = 0$ & At x = L $\Rightarrow \psi(x=L) = 0$

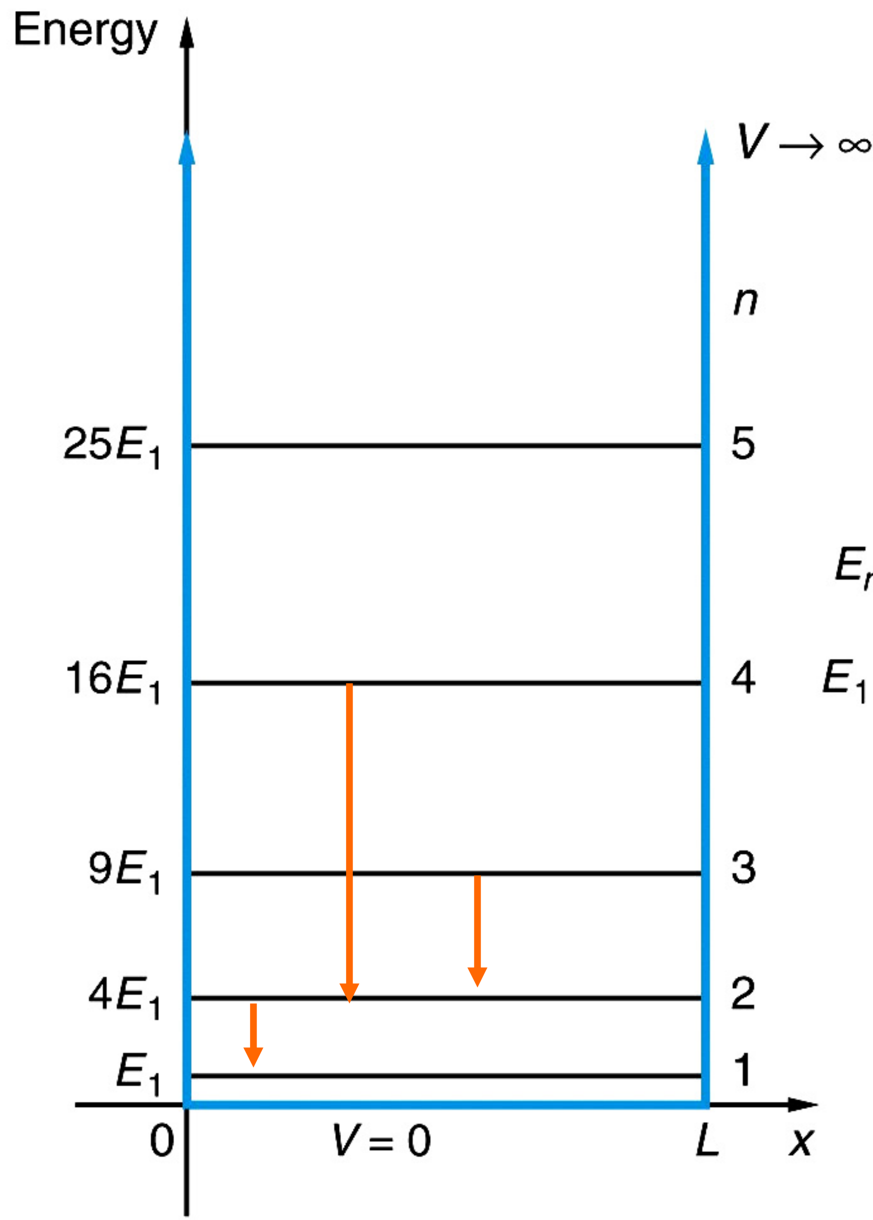
$\therefore\ \psi(x=0) = B = 0$ (Continuity condition at x =0)

& $\psi(x=L) = 0 \Rightarrow$ A Sin kL = 0 (Continuity condition at x =L)

$$\Rightarrow kL = n\pi \Rightarrow k = \frac{n\pi}{L}, n = 1,2,3,...\infty$$

So what does this say about Energy E ? : $E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}$ Quantized (not Continuous)!

# Quantized Energy levels of Particle in a Box




$$E_n = n^2 E_1$$

$$E_1 = \frac{\pi^2 \hbar^2}{2mL^2}$$

# What About the Wave Function Normalization ?

The particle's Energy and Wavefunction are determined by a number n

We will call n $\rightarrow$ Quantum Number , just like in Bohr's Hydrogen atom

What about the wave functions corresponding to each of these energy states?

$$\psi_n = A\sin(kx) = A\sin(\frac{n\pi x}{L}) \quad \text{for } 0<x<L$$

$$= 0 \quad \text{for } x \geq 0,\ x \geq L$$

Normalized Condition :

$$1 = \int_0^L \psi_n^* \psi_n dx = A^2 \int_0^L Sin^2(\frac{n\pi x}{L})$$

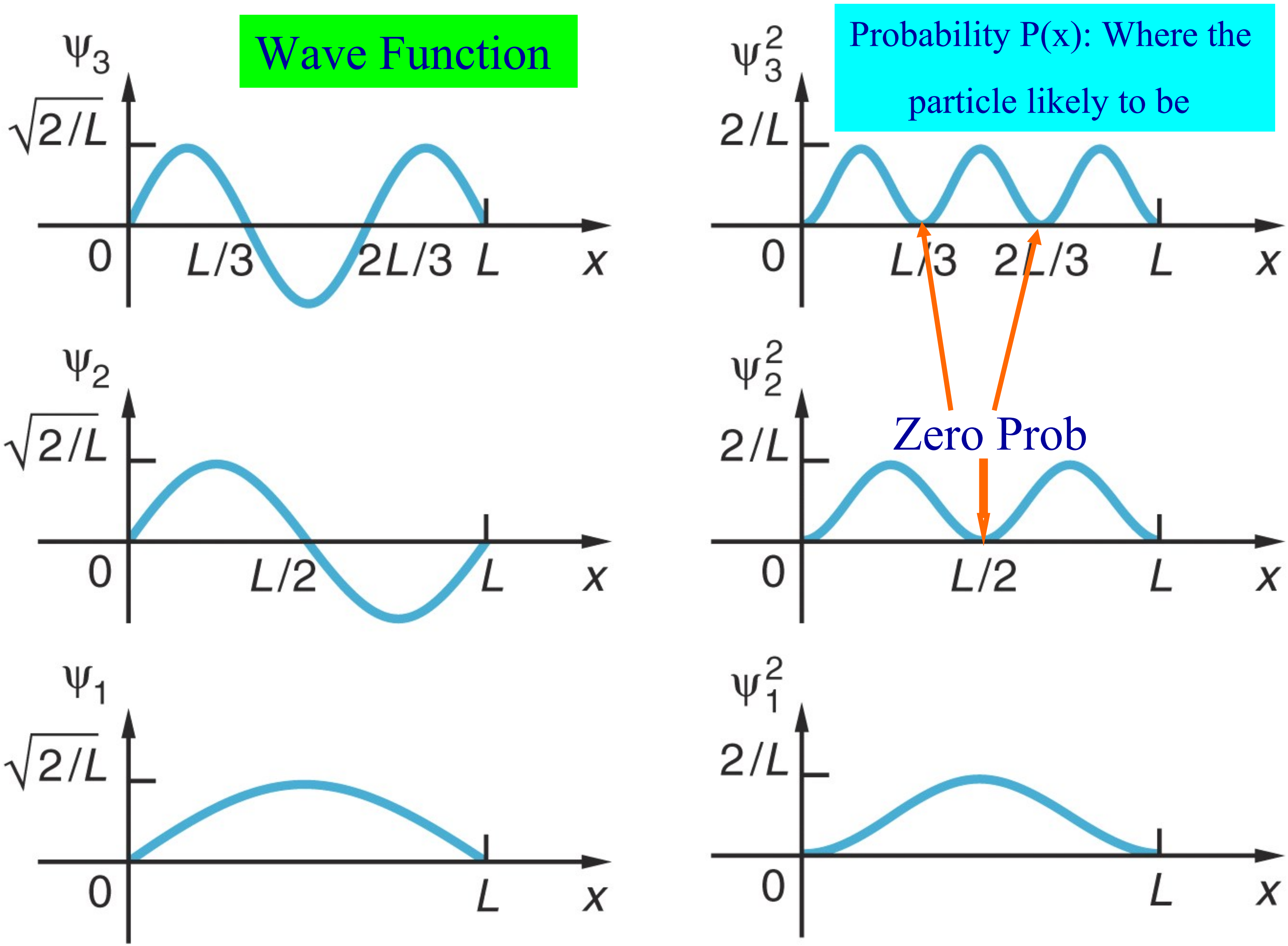
Use $2\text{Sin}^2\theta = 1 - 2Cos2\theta$

$$1 = \frac{A^2}{2}\int_0^L \left(1 - \cos(\frac{2n\pi x}{L})\right)$$ and since $\int \cos\theta = \sin\theta$

$$1 = \frac{A^2}{2}L \quad \Rightarrow A = \sqrt{\frac{2}{L}}$$

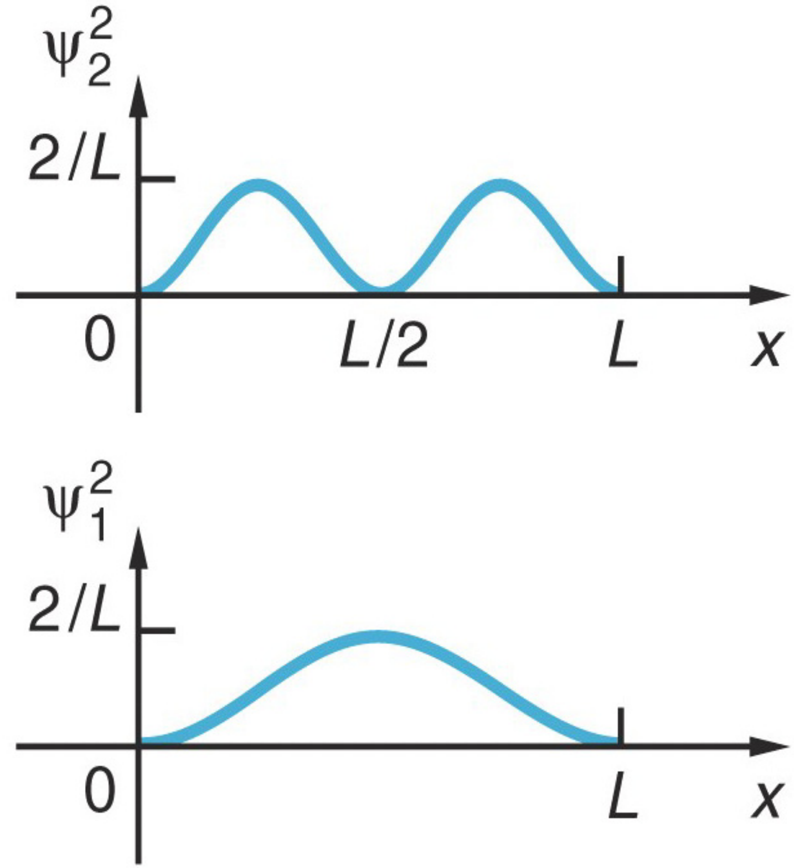
So $\psi_n = \sqrt{\frac{2}{L}}\sin(kx) = \sqrt{\frac{2}{L}}\sin(\frac{n\pi x}{L})$ ...What does this look like?

# Wave Functions : Shapes Depend on Quantum # n



# Where in The World is Carmen San Diego?

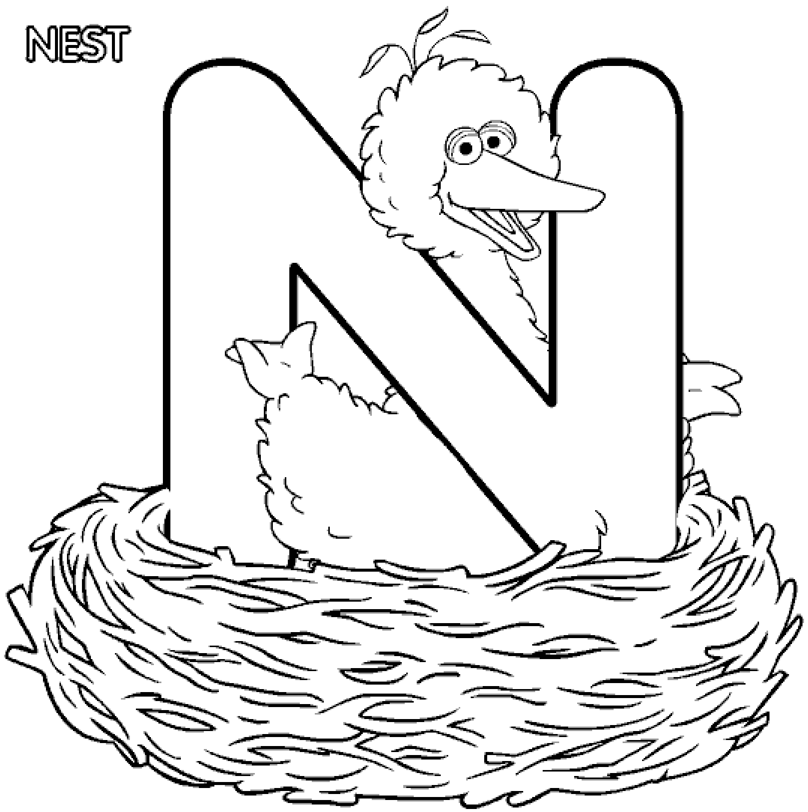
- We can only guess the probability of finding the particle somewhere in x
  - For n=1 (ground state) particle most likely at x = L/2
  - For n=2 (first excited state) particle most likely at L/4, 3L/4
    - Prob. Vanishes at x = L/2 & L
      - How does the particle get from just before x=L/2 to just after?
        - » QUIT thinking this way, particles don't have trajectories
        - » Just probabilities of being somewhere




Classically, where is the particle most Likely to be : Equal prob of being anywhere inside the Box

NOT SO says Quantum Mechanics!

# Remember Sesame Street ?




This particle in the box is brought to you by the letter

**n**

Its the Big Boss Quantum Number